The double Caldeira-Leggett model: Derivation and solutions of the master equations, reservoir-induced interactions and decoherence

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Abstract

In this paper we analyze the double Caldeira-Leggett model: the path integral approach to two interacting dissipative harmonic oscillators. Assuming a general form of the interaction between the oscillators, we consider two different situations: i) when each oscillator is coupled to its own reservoir, and ii) when both oscillators are coupled to a common reservoir. After deriving and solving the master equation for each case, we analyze the decoherence process of particular entanglements in the positional space of both oscillators. To analyze the decoherence mechanism we have derived a general decay function for the off-diagonal peaks of the density matrix, which applies both to a common and separate reservoirs. We have also identified the expected interaction between the two dissipative oscillators induced by their common reservoir. Such reservoir-induced interaction, which gives rise to interesting collective damping effects, such as the emergence of relaxation- and decoherence-free subspaces, is shown to be blurred by the high-temperature regime considered in this study. However, we find that different interactions between the dissipative oscillators, described by rotating or counter-rotating terms, result in different decay rates for the interference terms of the density matrix.

I. INTRODUCTION

At the beginning of the 1980s, the work of Zurek [1], Caldeira and Leggett (CL) [2], and Zeh and Joos [3] played a decisive role in the understanding of the still unsolved phenomenon of quantum measurement; more specifically, the collapse of the wave function and the associated decoherence of superposition states [4]. Taking the reservoir into account explicitly as a quantum ingredient, and analyzing its effect on the evolution of an initial pure state into a statistical mixture, these papers shed light on the shadowy interface between microscopic and macroscopic domains. Although the wave function collapse remains an obscure process, despite striking contributions also dating from the eighties [5], much is known today about the mechanisms leading to decoherence. In the last few decades we have analyzed this phenomenon exhaustively, enabling the proposition of a plethora of protocols to circumvent it, ranging from quantum error correction codes QECC [6] and engineered reservoirs [7] to dynamical decoupling [8] and relaxation- and decoherence-free subspaces (R-DFSs) [9, 10].

More recently, it was demonstrated that entanglement shows scaling behavior in the vicinity of the transition point [11]. This connection between the theories of critical phenomena and quantum information, together with the search for R-DFSs — which encompasses dissipative coupled systems — has triggered the study of fundamental quantum processes in the domain of many-body physics. Apart from the crucial role played by entanglements in the understanding of quantum phase transitions [12], the study of the complex dynamics of coherence and decoherence of superposition states in networks of dissipative quantum systems has also produced interesting results for quantum information theory [10, 13, 14]. In particular, in Ref. [10] a correlation function was introduced to provide the analytical conditions for the existence of R-DFSs in a system of interacting dissipative resonators. This correlation function measures the reliability of a decoherence-free subspace. Apart from the correlation function, Refs. [10, 13, 14] put forward, as a conceptual novelty, the need to consider distinct reservoirs for distinct quantum systems.

In this study, we consider a system of two interacting harmonic oscillators, in two situations: i) when each one is coupled to its own reservoir and ii) when both oscillators are coupled to a common reservoir. As argued in Ref. [10], the former case, where each system interacts with its own reservoir, is the most usual situation. Considering, for example, a network of coupled cavities, even when they have the same quality factor their damping

mechanisms are independent, except when they interact strongly [10]. By strong interaction it is meant that the coupling strength λ between the N cavities composing the network must satisfy the relation $N\lambda \sim \omega$, ω being the natural frequency of each cavity mode in the degenerate case, where $\omega_1 = \cdots = \omega_N = \omega$. In such a strong coupling limit, the results emerging in the case of a common reservoir are completely similar to those for the case where distinct reservoirs are assumed. Conversely, in the weak coupling limit, where $N\lambda \ll \omega$, separate reservoirs must be assigned to each cavity composing the network.

Even in particular cases where we could, in principle, assign a common reservoir to different quantum systems, such as a sample of atoms inside the same cavity — the multimodal cavity playing the role of a common reservoir — such a reservoir turns out to act as distinct reservoirs when the transition frequencies of the atoms are significantly far from each other. In this situation, each atom interacts with the reservoir modes around its own frequency transition, and the absence of (or small) intersection between the reservoir modes addressed by distinct atoms makes the common reservoir act as several distinct ones [10]. Evidently, when the atomic transition frequencies are not sufficiently apart from each other, the overlap between the reservoir modes addressed by each atom indicates that they start to interact through their "partially common reservoir". Only in the limiting case, where all the atoms in the sample have the same transition frequency — and a full overlap between the reservoir modes is achieved — is a "completely common reservoir" accessed by the whole atomic sample [10]. At this limit, the interaction between the atoms through their common reservoir is maximized, opening up the possibility of an interesting feature arising from interacting dissipative systems: the R-DFSs. Therefore, the assessment of such R-DFSs — which may become indispensable for the implementation of quantum information — requires a completely understanding of the dissipative mechanism for coupled quantum systems: either through a common or distinct reservoirs. Apart from the emergence of R-DFSs, the subject of collective damping effects has recently produced interesting results, such as the nonadditivity of decoherence rates observed in a network of dissipative oscillators [10, 13, 14], as well as in superconducting qubits [15]. Returning to the atomic samples, we stress that the collective damping effects coming from two-atom systems [16], can be directly identified with the nonadditivity of decoherence rates.

The problem of two coupled harmonic oscillators has already been discussed in the literature from various perspectives. We first mention a proposal, based on the possibility of

performing a reversible coupling between high-Q cavities, to achieve reversible decoherence of a mesoscopic superposition of field states [17]. A theoretical approach for such an experimental proposal is given in Ref. [18], where a common reservoir is assumed for both cavities. In Ref. [19] a system of two coupled cavities has been analyzed in which just one of the cavities interacts with a reservoir. A master equation is derived for the case of strongly coupled cavities and it is shown that the relaxation term is not simply the standard one, obtained by neglecting the interaction between the cavities. Finally, in Ref. [13], each cavity is assumed to interact with its own reservoir and a detailed investigation is carried out for both regimes of weakly and strongly coupled oscillators. In order that the oscillators interact only through their direct coupling and not indirectly through their couplings with a common reservoir, it is advisable to assume two distinct reservoirs. It is worth mentioning that a general treatment of a network of coupled dissipative harmonic oscillators has recently been presented [20] for any topology — i.e., irrespective of how the oscillators are coupled together, the strength of their couplings, and their natural frequencies. As in Refs. [10, 13, 14], the authors start with a general, more realistic, scenario where each oscillator is coupled to its own reservoir, and proceed later to the particular case where all the network oscillators are coupled to a common reservoir.

In the present paper, a general form for the interaction between two dissipative oscillators is considered and both situations, of distinct reservoirs and a common one, are analyzed. Moreover, instead of the master equation approach, we follow the path integral approach adopted by CL in their linear response model [2], with which considerable progress has been made on the subject of quantum dissipation in several areas of physics. In fact, with their linear response model, CL accounted for the influence of dissipation on quantum tunneling in macroscopic systems [21]. Quantum Brownian motion has also been approached through the influence-functional method [2, 22, 23]. Moreover, the linear response model in [2] has been applied to many topics in solid state physics, for example the dynamics of polarons [24] and a particle coupled to a Luttinger liquid [25].

Long before the CL model, functional integral calculations were used by Feynman to analyze the problem of polarons in a polar crystal [26] and, more recently, they have been applied to the problem of bipolarons [27]. As a direct extension of the Feynman polaron model, in the path-integral approach to the bipolaron each electron is harmonically coupled to a fictitious heavy particle which replaces the virtual phonon cloud. Each electron also

interacts with the fictitious particle of the other electron, apart from the Coulomb repulsion between the two. Therefore, the double CL model presented here bears some resemblance to the bipolaron problem, with our oscillators (reservoirs) replacing the electrons (fictitious particles). By replacing the interaction between the two dissipative oscillators by the Coulomb repulsion between the electrons, we end up with a dissipative bipolaron-type model. In fact, as we demonstrate here, the oscillators interacts indirectly through their common reservoir or even their separate ones.

Before closing this Introduction, we must mention the recent result in Ref. [28], where the authors point out the effective coupling that is induced between two Brownian noninteracting particles by a common reservoir. Such an effective coupling depends on the choice made for the spectral function of the reservoir. In the present study, working with an ohmic reservoir at the high-temperature limit, we find that this induced effective coupling occurs in the case of a common reservoir while, as expected, it is absent in the case of distinct reservoirs. We also note that such an effective coupling induced by a common reservoir is also pointed out in Refs. [10, 13, 14, 20].

Summarizing, in the present paper we employ the path integral approach to treat a network of two interacting dissipative harmonic oscillators. Assuming a general form of the interaction between the oscillators, we consider two different situations: i) when each oscillator is coupled to its own reservoir, and ii) when both oscillators are coupled to a common reservoir. We derive and solve the master equation for each case and, in the latter, we identify the reservoir-induced coupling between the oscillators, which arises even when the original interaction between them is schwitched off. We verify that such a reservoir-induced coupling encompasses both dissipative and diffusive terms which couple together the variables of both oscillators. These terms thus account for the energy loss of the oscillators through each other, apart from a joint diffusive process. From the solutions we have found for the master equations we also compute a general expression for the decay rate of the off-diagonal peaks of the density matrix of initial superposition states, which also applies to both cases of a common and distinct reservoirs. Finally, considering different interactions between the oscillators, we analyze the decoherence process of particular entanglements in their positional spaces.

II.   THE DOUBLE CALDEIRA-LEGGETT MODEL

The CL problem consists in applying the influence-functional method of Feynman and Vernon to study the dissipation-fluctuation dynamics of a quantum system (S) interacting with a reservoir (R) modelled by a collection of one-dimensional harmonic oscillators. The CL Hamiltonian is given by $H_{CL} = H_S + H_R + H_{S/R}$, where the Hamiltonian of the system, represented by a particle of mass m and coordinates q and p, is given by

$$H_S = \frac{p^2}{2m} + V(q); \tag{1}$$

the Hamiltonian for the reservoir, consisting of a collection of harmonic oscillators $\{\omega_k\}$ of coordinates $\{q_k\}$ and $\{p_k\}$, and masses $\{m_k\}$ (the subscript indicating the kth oscillator reservoir) reads

$$H_R = \frac{1}{2} \sum_k \left(\frac{p_k^2}{m_k} + m_k \omega_k^2 q_k^2 \right); \tag{2}$$

and the interaction Hamiltonian, linear by hypothesis and defined by the coupling constants C_k, is given by

$$H_{S/R} = q \sum_k C_k q_k. \tag{3}$$

The double CL model considered here consists of two quantum systems S_1 and S_2, of masses m_1 and m_2, coupled through the general form

$$H_{S_1/S_2} = \lambda_{11} q_1 q_2 + \lambda_{12} q_1 p_2 + \lambda_{21} q_2 p_1 + \lambda_{22} p_1 p_2. \tag{4}$$

Two different situations arise, however, in the system-reservoir interactions: i) one that seems more appropriate for most physical systems, where each oscillator is coupled to its own reservoir, and ii) another, which in practice is rather unusual, where both oscillators are coupled to a common reservoir. The Hamiltonians governing the evolutions of the two coupled dissipative systems are given, respectively, by

$$H_i = \sum_\ell \left(H_{S_\ell} + H_{S_\ell/R_\ell} + H_{R_\ell} \right) + H_{S_1/S_2}, \tag{5a}$$

$$H_{ii} = \sum_\ell \left(H_{S_\ell} + H_{S_\ell/R} \right) + H_R + H_{S_1/S_2}, \tag{5b}$$

where $\ell, \ell', \ell'' = 1, 2$ from here on.

In the context of a network of cavities coupled by superconducting waveguides [17, 29, 30], distinct reservoirs must be assumed, in general, for distinguishable cavities, even if they

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exhibit equal quality factors, as long as there are no correlations whatsoever between the reservoirs. The same applies to distinguishable trapped ions or a traveling field reaching distinguishable optical elements. However, as demonstrated in Ref. [10], a sufficiently strong coupling between the cavities or the trapped ions lead to a correlation between the reservoirs since, as expected, each particular system of the network start to interact with all the reservoirs. There are a few particular situations where a set of quantum systems may interact with a common reservoir, such as an atomic sample or distinct fields inside a perfect closed cavity. In the former case, different atomic transitions couple with different reservoir modes and again, the correlations between these reservoir modes define either a common or distinct reservoirs, as will be demonstrated below. In the latter, it has been showed that the proximity of the distinct field modes sets the strength of the correlation function between the reservoir modes, which governs the emergence of both R-DFSs [10].

A.   Distinct Reservoirs

Starting with the case of distinct reservoirs, the term of the Hamiltonian accounting for the oscillators and their interactions with the reservoirs is given by

$$H_i = \sum_\ell \left[\frac{p_\ell^2}{2m_\ell} + V_\ell(q_\ell) + \sum_k \left(\frac{p_{\ell k}^2}{2m_{\ell k}} + \frac{m_{\ell k}}{2}\omega_{\ell k}^2 q_{\ell k}^2 + C_{\ell k}q_\ell q_{\ell k} \right) \right] + H_{S_1/S_2}. \tag{6}$$

The Lagrangian associated with Hamiltonian H_i, which defines the action $\mathcal{S} = \int_0^t \mathcal{L}_i \, d\tau$ of the Feynman-Vernon theory, is given by

$$\mathcal{L}_i = \sum_\ell \left\{ \frac{\mu_\ell \dot{q}_\ell^2}{2} - V_\ell(q_\ell) + \sum_k \left[\frac{m_{\ell k}}{2} \left(\dot{q}_{\ell k}^2 - \omega_{\ell k}^2 q_{\ell k}^2 \right) - C_{\ell k}q_\ell q_{\ell k} \right] \right\} - \mathcal{L}, \tag{7}$$

where $\mathcal{L}$, the Lagrangian associated with the interaction between the two systems, reads

$$\mathcal{L} = \lambda_{11}q_1 q_2 + \lambda_{21}\mu_{11}q_2 \left(\dot{q}_1 - \frac{\lambda_{21}}{2}q_2 \right) + \lambda_{12}\mu_{22}q_1 \left(\dot{q}_2 - \frac{\lambda_{12}}{2}q_1 \right) + \mu_{12} \left(\dot{q}_1 - \lambda_{21}q_2 \right) \left(\dot{q}_2 - \lambda_{12}q_1 \right), \tag{8}$$

in which

$$\mu_{\ell \ell'} = \frac{m_\ell \left(\lambda_{22}m_{\ell'} \right)^{1-\delta_{\ell \ell'}}}{1 - \lambda_{22}^2 m_1 m_2} \tag{9}$$

stands for reduced masses which, remarkably, arise exclusively from the $p_1 p_2$ coupling between the systems.

B. A Common Reservoir

When a common reservoir is assumed, the term of the Hamiltonian accounting for both oscillators and their interaction with the reservoir becomes

$$H_{ii} = \sum_{\ell} \left(\frac{p_\ell^2}{2m_\ell} + V_\ell(q_\ell) + \sum_k C_{\ell k} q_\ell q_k \right) + \frac{1}{2} \sum_k \left(\frac{p_k^2}{m_k} + m_k \omega_k^2 q_k^2 \right) + H_{S_1/S_2}. \tag{10}$$

The Lagrangian following from Hamiltonian H_{ii}, is given by

$$\mathcal{L}_{ii} = \sum_{\ell} \left(\frac{\mu_\ell \dot{q}_\ell^2}{2} - V_\ell(q_\ell) - \sum_k C_{\ell k} q_\ell q_k \right) + \sum_k \frac{m_k}{2} \left(\dot{q}_k^2 - \omega_k^2 q_k^2 \right) - \mathcal{L} \tag{11}$$

where $\mathcal{L}$ is as defined in Eq. (8).

Before proceeding to the calculations of the propagator for the double CL model through the influence-functional method of Feynman and Vernon, we first diagonalize the Hamiltonian describing both coupled systems

$$H_{S_1+S_2} = \sum_{\ell} \left(\frac{p_\ell^2}{2m_\ell} + V_\ell(q_\ell) \right) + H_{S_1/S_2}. \tag{12}$$

This diagonalization is indispensable to define the strength of the interaction parameters $\{\lambda_{\ell\ell'}\}$ which result in positive values for the normal-mode frequencies. Otherwise, we could have started from a positive-definite Hamiltonian with a lower bound for the energy spectrum [13, 14, 20, 31, 32].

III. DIAGONALIZATION OF THE COUPLED SYSTEMS

Assuming, from here on, that the coupled systems S_1 and S_2 are harmonic oscillators of frequencies ω_1 and ω_2, the diagonalized Hamiltonian is described in terms of unitary masses and the normal-mode coordinates Q_ℓ and P_ℓ, defined in Eqs. (95), as

$$H_{S_1+S_2} = \frac{1}{2} \sum_{\ell} \left(P_\ell^2 + \Omega_\ell^2 Q_\ell^2 \right). \tag{13}$$

The original masses m_ℓ have been absorbed by the normal-mode frequencies Ω_ℓ^2, obtained in Eq. (88), which account for the effective interactions g_ℓ, given in Eq. (87), apart from the natural frequencies ω_1 and ω_2. With the condition that the normal-mode frequencies Ω_ℓ

assume positive values, it is straightforward to show that the relations

$$\left(\frac{\omega_1 - \omega_2}{2}\right)^2 + |g_2|^2 \geq |g_1|^2 \left(\frac{\omega_1 - \omega_2}{\omega_1 + \omega_2}\right)^2 , \tag{14a}$$

$$\left(|g_2|^2 - |g_1|^2\right)^2 + (\omega_1\omega_2)^2 \geq 2\omega_1\omega_2 \left(|g_1|^2 + |g_2|^2\right) , \tag{14b}$$

must be satisfied.

The coordinates Q_ℓ and P_ℓ, associated with the normal-mode frequencies, follow from the previous generalized coordinates q_ℓ and p_ℓ, as described by the transformation (91). With these normal-mode frequencies and coordinates, the full Hamiltonians H_i and H_{ii} becomes:

$$H_i = \frac{1}{2}\sum_\ell \left(P_\ell^2 + \Omega_\ell^2 Q_\ell^2 + 2\sqrt{\frac{2\hbar}{m_\ell\omega_\ell}}\sum_{\ell'}[\mathrm{Re}\,(c_{\ell\ell'}Q_{\ell'}) + \mathrm{Im}\,(d_{\ell\ell'}P_{\ell'})]\sum_k C_{\ell k}q_k\right)$$
$$+ \frac{1}{2}\sum_k \left(\frac{p_k^2}{m_k} + m_k\omega_k^2 q_k^2\right) , \tag{15a}$$

$$H_{ii} = \frac{1}{2}\sum_\ell \left[P_\ell^2 + \Omega_\ell^2 Q_\ell^2 + \sum_k\left(\frac{p_{\ell k}^2}{m_{\ell k}} + m_{\ell k}\omega_{\ell k}^2 q_{\ell k}^2\right.\right.$$
$$\left.\left.+ 2\sqrt{\frac{2\hbar}{m_\ell\omega_\ell}}\sum_{\ell'}[\mathrm{Re}\,(c_{\ell\ell'}Q_{\ell'}) + \mathrm{Im}\,(d_{\ell\ell'}P_{\ell'})]C_{\ell k}q_{\ell k}\right)\right] , \tag{15b}$$

where we have used the inverse of the transformation Eq. (91), given by

$$q_\ell = \sqrt{\frac{2\hbar}{m_\ell\omega_\ell}}\sum_{\ell'}[\mathrm{Re}\,(c_{\ell\ell'}Q_{\ell'}) + \mathrm{Im}\,(d_{\ell\ell'}P_{\ell'})] , \tag{16a}$$

$$p_\ell = \sqrt{2\hbar m_\ell\omega_\ell}\sum_{\ell'}[\mathrm{Re}\,(d_{\ell\ell'}P_{\ell'}) - \mathrm{Im}\,(c_{\ell\ell'}Q_{\ell'})] , \tag{16b}$$

with the coefficients

$$c_{\ell\ell'} = \mathcal{N}_{\ell'}\sqrt{\frac{\Omega_{\ell'}}{2\hbar}}[-\Delta_{1\ell}(\Omega_{\ell'}) + \Delta_{2\ell}(\Omega_{\ell'})]\,\mathrm{e}^{i\phi_\ell} , \tag{17a}$$

$$d_{\ell\ell'} = \mathcal{N}_{\ell'}\sqrt{\frac{1}{2\hbar\Omega_{\ell'}}}[\Delta_{1\ell}(\Omega_{\ell'}) + \Delta_{2\ell}(\Omega_{\ell'})]\,\mathrm{e}^{i\phi_\ell} , \tag{17b}$$

and the functions $\Delta_{\ell\ell'}(\Omega_{\ell'})$ given by Eq. (90b).

Therefore, through the diagonalized form of the Hamiltonian $H_{S_1+S_2}$, we get two independent and generalized CL Hamiltonians in terms of normal coordinates Q_ℓ and P_ℓ. The Hamiltonians (15a) and (15b) describe two independent harmonic oscillators, both interacting with a common reservoir in the latter case and each one interacting with its own reservoir in the former. By a generalized CL Hamiltonian we mean that the system-reservoir coupling

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exhibits, due to the general form of the interaction between the two oscillators in Eq. (5), a momentum-position term apart from the usual position-position one. Thus, differently from the CL model, where a momentum-position coupling is easily handled through a coordinate transformation [2], in the present double CL model this momentum-position coupling hampers the application of the influence-functional method, since the derived Lagrangian involves tricky reservoir-reservoir interaction terms. Therefore, in spite of the diagonalization of the interaction described by Hamiltonian $H_{S_1+S_2}$, it is preferable to approach the problem, for both distinct reservoirs and a common one, through the Lagrangians in Eqs. (7) and (11), respectively. As demonstrated below, in the former case we end up with a product of two influence functionals, identical to that of the CL model, whereas for a common reservoir the influence functional does not factorize into individual functionals.

IV. FEYNMAN-VERNON THEORY

To obtain the reduced master equation describing the time evolution of the coupled systems S_1 and S_2, we proceed from the integral form of the density operator of the whole system at a time t, given by $\rho(t) = U(t)\rho(0)U^\dagger(t)$, where the evolution operator $U(t)$ follows from Hamiltonians H_i and H_{ii}. Next, we assign the coordinates x_ℓ and y_ℓ to system S_ℓ and the N-component vectors $\mathbf{X}_\ell = (X_{\ell 1}, \ldots, X_{\ell N})$ and $\mathbf{Y}_\ell = (Y_{\ell 1}, \ldots, Y_{\ell N})$ to reservoirs R_ℓ. In the case of a common reservoir, we only have to disregard one of the reservoirs R_ℓ, in the expression derived for the case of distinct reservoirs, to obtain the associated influence functional. Using the notation $\{x_\ell\} = x_1, x_2$ and $\{\mathrm{d}\, x_\ell\} = \mathrm{d}\, x_1\, \mathrm{d}\, x_2$, and the same form for other variables, we obtain the matrix element in the coordinate representation

$$
\begin{aligned}
&\langle \{x_\ell\}, \{\mathbf{X}_\ell\} \,|\rho(t)|\, \{y_\ell\}, \{\mathbf{Y}_\ell\}\rangle \\
&= \int \mathrm{d}\,\{x'_\ell\}\, \mathrm{d}\,\{y'_\ell\}\, \mathrm{d}\,\{\mathbf{X}'_\ell\}\, \mathrm{d}\,\{\mathbf{Y}'_\ell\}\ \ \langle \{x'_\ell\}, \{\mathbf{X}'_\ell\} \,|\rho(0)|\, \{y'_\ell\}, \{\mathbf{Y}'_\ell\}\rangle \\
&\quad \times K\left(\{x_\ell\}, \{\mathbf{X}_\ell\}, t; \{x'_\ell\}, \{\mathbf{X}'_\ell\}, 0\right) K^*\left(\{y_\ell\}, \{\mathbf{Y}_\ell\}, t; \{y'_\ell\}, \{\mathbf{Y}'_\ell\}, 0\right),
\end{aligned}
\tag{18}
$$

where functional integrations are evaluated over paths $x_\ell(t')$, $y_\ell(t')$, $\mathbf{X}_\ell(t')$ and $\mathbf{Y}_\ell(t')$, with endpoints $x_\ell(t) = x_\ell$, $x_\ell(0) = x'_\ell$, $y_\ell(t) = y_\ell$, $y_\ell(0) = y'_\ell$, $\mathbf{X}_\ell(t) = \mathbf{X}_\ell$, $\mathbf{X}_\ell(0) = \mathbf{X}'_\ell$, $\mathbf{Y}_\ell(t) = \mathbf{Y}_\ell$, and $\mathbf{Y}_\ell(0) = \mathbf{Y}'_\ell$. The propagator K is given by

$$
K\left(\{x_\ell\}, \{\mathbf{X}_\ell\}, t; \{x'_\ell\}, \{\mathbf{X}'_\ell\}, 0\right) = \int \mathrm{D}\,\{x_\ell\}\, \mathrm{D}\,\{\mathbf{X}_\ell\}\ \exp\left(\frac{i}{\hbar}\mathcal{S}\left[\{x_\ell\}, \{\mathbf{X}_\ell\}\right]\right).
\tag{19a}
$$

The action $\mathcal{S}$ follows from the Lagrangian $\mathcal{L} = \mathcal{L}_i$ or $\mathcal{L}_{ii}$, as $\mathcal{S} = \int_0^t \mathcal{L} \, \mathrm{d} t'$. Tracing out the reservoir coordinates, we obtain the reduced density operator describing the coupled systems S_1 and S_2 under the influence of their respective reservoir, given by

$$\widetilde{\rho}\left(\{x_\ell\}, \{y_\ell\}, t\right) = \int \mathrm{d}\left\{x_\ell'\right\} \mathrm{d}\left\{y_\ell'\right\} \ \mathrm{J}\left(\{x_\ell\}, \{y_\ell\}, t; \{x_\ell'\}, \{y_\ell'\}, 0\right) \widetilde{\rho}\left(\{x_\ell'\}, \{y_\ell'\}, 0\right), \quad (20)$$

where we have assumed that the system-reservoir coupling is turned on suddenly, such that the total density operator is initially given by $\rho(0) = \widetilde{\rho}(0)\rho_{R_1+R_2}(0)$. The propagator for the density operator turns out to be

$$\mathrm{J}\left(\{x_\ell\}, \{y_\ell\}, t; \{x_\ell'\}, \{y_\ell'\}, 0\right)$$
$$= \int \mathrm{D}\left\{x_\ell\right\} \mathrm{D}\left\{y_\ell\right\} \ \mathcal{F}\left[\{x_\ell\}, \{y_\ell\}\right]$$
$$\times \exp\left\{\frac{i}{\hbar}\left[\sum_\ell \left(\mathcal{S}_{S_\ell}\left[\{x_\ell\}\right] - \mathcal{S}_{S_\ell}\left[\{y_\ell\}\right]\right) + \mathcal{S}_{S_1/S_2}\left[\{x_\ell\}\right] - \mathcal{S}_{S_1/S_2}\left[\{y_\ell\}\right]\right]\right\}. \quad (21)$$

Although expression (20) applies to both cases, i) and ii), the computation of the influence functional $\mathcal{F}\left[\{x_\ell\}, \{y_\ell\}\right]$ [33, 34] representing the effects of the reservoirs on the systems is completely different for the two cases.

A. Distinct Reservoirs

When considering distinct reservoirs, the influence functional in Eq. (21) simply factorizes as

$$\mathcal{F}\left[\{x_\ell\}, \{y_\ell\}\right] = \prod_\ell \mathcal{F}\left[x_\ell, y_\ell\right], \quad (22)$$

and the component arising from the interaction of system S_ℓ with reservoir R_ℓ, is written as

$$\mathcal{F}\left[x_\ell, y_\ell\right] = \int \mathrm{d}\,\mathbf{X}_\ell \, \mathrm{d}\,\mathbf{X}_\ell' \, \mathrm{d}\,\mathbf{Y}_\ell' \ \rho_{R_\ell}\left(\mathbf{X}_\ell', \mathbf{Y}_\ell', 0\right) \int \mathrm{D}\,\mathbf{X}_\ell \, \mathrm{D}\,\mathbf{Y}_\ell$$
$$\times \exp\left\{\frac{i}{\hbar}\left(\mathcal{S}_{R_\ell}\left[\mathbf{X}_\ell\right] - \mathcal{S}_{R_\ell}\left[\mathbf{Y}_\ell\right] + \mathcal{S}_{S_\ell/R_\ell}\left[x_\ell, \mathbf{X}_\ell\right] - \mathcal{S}_{S_\ell/R_\ell}\left[y_\ell, \mathbf{Y}_\ell\right]\right)\right\} \quad (23)$$

Evidently, when disregarding the interaction between the systems, described by Hamiltonian H_{S_1/S_2} in Eq. (4), the propagator J in Eq. (21) reduces to a product of propagators identical to that obtained in the CL model. However, the influence functional factorizes independently of any requirement for non-interacting systems, since it only takes into account the interaction between the systems and their respective reservoirs. Therefore,

the influence functional (22) is obtained directly as a product of the functional obtained from the CL model. All that remains to be done is to obtain the propagator (21) by considering the actions coming from the interacting systems.

Thus, from the CL model we obtain directly the form

$$\mathcal{F}[x_\ell, y_\ell] = \exp\left\{-\frac{1}{\hbar}\int_0^t d\tau \int_0^\tau dt' \, \alpha_{\ell R}(\tau - t')\left[x_\ell(\tau) - y_\ell(\tau)\right]\left[x_\ell(t') - y_\ell(t')\right]\right\}$$
$$\times \exp\left\{-\frac{i}{\hbar}\int_0^t d\tau \int_0^\tau dt' \, \alpha_{\ell I}(\tau - t')\left[x_\ell(\tau) - y_\ell(\tau)\right]\left[x_\ell(t') + y_\ell(t')\right]\right\}, \quad (24)$$

with the real and imaginary parts of a function $\alpha(t)$ given by

$$\alpha_{\ell R}(t) = \sum_k \frac{C_{\ell k}^2}{2m_{\ell k}\omega_{\ell k}}\coth(\omega_{\ell k}\hbar\beta_\ell/2)\cos[\omega_{\ell k}(t)], \quad (25a)$$

$$\alpha_{\ell I}(t) = -\sum_k \frac{C_{\ell k}^2}{2m_{\ell k}\omega_{\ell k}}\sin[\omega_{\ell k}(t)]. \quad (25b)$$

Assuming that the reservoir modes are sufficiently closely spaced to allow a continuum summation, we define the spectral functions [2]

$$\chi_\ell(\omega) = \pi \sum_k \frac{C_{\ell k}^2}{2m_{\ell k}\omega_{\ell k}}\delta(\omega - \omega_{\ell k}). \quad (26)$$

The introduction of a frequency cutoff Ω_ℓ^C considerably higher than the characteristic frequencies of the problem, together with the assumption of an Ohmic reservoir, where the distributions $\chi_\ell(\omega)$ are defined by the damping constants η_ℓ, such that

$$\chi_\ell(\omega) = \begin{cases} \eta_\ell\omega, & \omega < \Omega_\ell^C, \\ 0, & \omega > \Omega_\ell^C, \end{cases} \quad (27)$$

enable us to rewrite Eq. (25) as

$$\alpha_{\ell R}(t) = \frac{1}{\pi}\int_0^{\Omega_\ell^C} d\omega_\ell \, \eta_\ell\omega_\ell \coth(\omega_\ell\hbar\beta/2)\cos[\omega_\ell(t)], \quad (28a)$$

$$\alpha_{\ell I}(t) = -\frac{1}{\pi}\int_0^{\Omega_\ell^C} d\omega_\ell \, \eta_\ell\omega_\ell \sin[\omega_\ell(t)]$$
$$= \frac{\eta_\ell}{\pi t}\left(\Omega_\ell^C \cos[\Omega_\ell^C(t)] - \frac{\sin[\Omega_\ell^C(t)]}{t}\right). \quad (28b)$$

As we are interested in times much longer than the typical value $1/\Omega_\ell^C$, it follows that $\sin[\Omega_\ell^C(t)]/\pi t \approx \delta(t)$ and, consequently

$$\int_0^t d\tau \, \frac{\sin[\Omega_\ell^C(\tau)]}{\pi\tau}\left[x_\ell(\tau) - y_\ell(\tau)\right] \approx 0, \quad (29a)$$

$$\int_0^\tau dt \, [\dot{x}_\ell(t) + \dot{y}_\ell(t)]\frac{\sin[\Omega_\ell^C(\tau - t)]}{\pi(\tau - t)} \approx \frac{\dot{x}_\ell(\tau) + \dot{y}_\ell(\tau)}{2}. \quad (29b)$$

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With these approximations, the propagator becomes

$$
\mathrm{J}\left(\{x_\ell\},\{y_\ell\},t;\{x'_\ell\},\{y'_\ell\},0\right)
$$

$$
= \exp\left[-i\sum_\ell \frac{\gamma_\ell \mu_{\ell\ell}}{2\hbar}\left(x_\ell^2 - x_\ell'^2 - y_\ell^2 + y_\ell'^2\right)\right]\int \mathrm{D}\left\{x_\ell\right\}\mathrm{D}\left\{y_\ell\right\}
$$

$$
\times \exp\left\{\frac{i}{\hbar}\left[\sum_\ell\left(\widetilde{\mathcal{S}}_{S_\ell}[x_\ell] - \widetilde{\mathcal{S}}_{S_\ell}[y_\ell]\right) + \mathcal{S}_{S_1/S_2}\left[\{x_\ell\}\right] - \mathcal{S}_{S_1/S_2}\left[\{y_\ell\}\right]\right]\right\}
$$

$$
\times \exp\left\{-\sum_\ell \frac{\gamma_\ell\mu_{\ell\ell}}{\hbar}\left[\frac{2}{\pi}\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}t'\ \left[x_\ell(\tau) - y_\ell(\tau)\right]\left[x_\ell(t') - y_\ell(t')\right]\right.\right.
$$

$$
\times \int_0^{\Omega_\ell^C}\mathrm{d}\omega_\ell\ \omega_\ell\coth\left(\omega_\ell\hbar\beta_\ell/2\right)\cos\left[\omega_\ell(\tau - t')\right]
$$

$$
\left.\left.-i\int_0^t \mathrm{d}\tau\ \left[x_\ell(\tau)\dot{y}_\ell(\tau) - y_\ell(\tau)\dot{x}_\ell(\tau)\right]\right]\right\}, \tag{30}
$$

where we have defined the relaxation constant $\gamma_\ell = \eta_\ell/2\mu_{\ell\ell}$ for the two system-reservoir couplings and the renormalized actions

$$
\widetilde{\mathcal{S}}_{S_\ell}[x_\ell] = \int_0^t \mathrm{d}\tau\ \mathcal{L}_i(x_\ell,\dot{x}_\ell,t)
$$

$$
= \int_0^t \mathrm{d}\tau\ \left[\frac{\mu_{\ell\ell}}{2}\dot{x}_\ell^2 - \widetilde{V}_\ell(x_\ell)\right], \tag{31a}
$$

$$
\mathcal{S}_{S_1/S_2}\left[\{x_\ell\}\right] = \int_0^t \mathrm{d}\tau\ L_{12}\left(\{x_\ell\},\{\dot{x}_\ell\},t'\right)
$$

$$
= -\int_0^t \mathrm{d}\tau\ \left[\lambda_{11}x_1 x_2 + \mu_{12}\left(\dot{x}_1 - \lambda_{21}x_2\right)\left(\dot{x}_2 - \lambda_{12}x_1\right)\right.
$$

$$
\left.+\lambda_{12}\mu_{22}x_1\left(\dot{x}_2 - \frac{\lambda_{12}}{2}x_1\right) + \lambda_{21}\mu_{11}x_2\left(\dot{x}_1 - \frac{\lambda_{21}}{2}x_2\right)\right], \tag{31b}
$$

with similar expressions for the variables y_ℓ. The renormalized potentials $\widetilde{V}_\ell(x_\ell) = m_\ell\left[\omega_\ell^2 - (\Delta\omega_\ell)^2\right]x_\ell^2/2$, follow from the system-reservoir couplings which induce the shifts $(\Delta\omega_\ell)^2 = 2\eta_\ell\Omega_\ell^C/\pi m_\ell$. Thus, we can define a renormalized frequency, given by $\widetilde{\omega}_\ell^2 = \omega_\ell^2 - (\Delta\omega_\ell)^2$[2].

From the result that the functional integral for an infinitesimal time evolution can be approximated by [33],

$$
\int \mathrm{D}q\exp\left(\frac{i}{\hbar}\mathcal{S}\right) \approx \frac{1}{N}\exp\left(\frac{i}{\hbar}\mathcal{S}\right), \tag{32}
$$

we consider the evolution of the reduced density operator $\widetilde{\rho}$ on the infinitesimal time interval

between t and $t + \varepsilon$ ($\varepsilon \to 0$), proceeding from Eq. (19), to obtain

$$\widetilde{\rho}\left(\{x_\ell\}, \{y_\ell\}, t + \varepsilon\right)$$
$$= \int d\{x'_\ell\} d\{y'_\ell\} \ \ J\left(\{x_\ell\}, \{y_\ell\}, t + \varepsilon; \{x'_\ell\}, \{y'_\ell\}, t\right) \widetilde{\rho}\left(\{x'_\ell\}, \{y'_\ell\}, t\right). \tag{33}$$

We also assume the high-temperature limit $k_B T_\ell \gg \hbar\omega_\ell$ (for both reservoir frequencies $\omega_\ell \ll \Omega_\ell^C$), which allows analytical solutions of the integrals (in the variable ω_ℓ) when defining the propagator J and using, for any function F, the approximations $\dot{x}_\ell \approx \beta_{1\ell}/\varepsilon$, $\dot{y}_\ell \approx \beta_{2\ell}/\varepsilon$, and

$$\int_t^{t+\varepsilon} d\tau \, F\left[x_\ell(\tau)\right] \approx \varepsilon \, F\left[\frac{x_\ell + x'_\ell}{2}\right], \tag{34}$$

where we have defined the variables $\beta_{1\ell} = x_\ell - x'_\ell$ and $\beta_{2\ell} = y_\ell - y'_\ell$. With the above approximations, we obtain from Eq. (30) the propagator

$$J\left(\{x_\ell\}, \{y_\ell\}, t + \varepsilon; \{x_\ell - \beta_{1\ell}\}, \{y_\ell - \beta_{2\ell}\}, t\right)$$

$$= \mathcal{N} \exp\left\{ \frac{i}{\hbar} \left[\frac{\mu_{12}}{\varepsilon} \sum_\ell (-1)^\ell \beta_{\ell 1} \beta_{\ell 2} + \varepsilon \left(\lambda_{11} + \lambda_{12}\lambda_{21}\mu_{12}\right) \left[\prod_\ell \left(y_\ell - \frac{\beta_{2\ell}}{2}\right) - \prod_\ell \left(x_\ell - \frac{\beta_{1\ell}}{2}\right) \right] \right.\right.$$

$$+ \sum_{\ell, \neq \ell'} \lambda_{\ell\ell'} \left\{ \sum_{\ell''} (-1)^{\ell'+\ell''} \mu_{\ell''\ell'} \left[\beta_{2\ell''} \left(y_\ell - \frac{\beta_{2\ell}}{2}\right) - \beta_{1\ell''} \left(x_\ell - \frac{\beta_{1\ell}}{2}\right) \right] \right.$$

$$\left.\left.+ \frac{\varepsilon}{2} \lambda_{\ell\ell'} \mu_{\ell'\ell'} \left[\left(x_\ell - \frac{\beta_{1\ell}}{2}\right)^2 + \left(y_\ell - \frac{\beta_{2\ell}}{2}\right)^2 \right] \right\} \right] \right\}$$

$$\times \prod_\ell \exp\left\{ \frac{i\mu_{\ell\ell}}{\hbar} \left[\frac{1}{2} \left(\beta_{1\ell}^2 - \beta_{2\ell}^2\right) \left(\gamma_\ell + \frac{1}{\varepsilon}\right) - \gamma_\ell \left(x_\ell - y_\ell\right) \left(\beta_{1\ell} + \beta_{2\ell}\right) \right] \right.$$

$$\left. - \frac{i\varepsilon}{\hbar} \left[\widetilde{V}_\ell\left(x_\ell - \frac{\beta_{1\ell}}{2}\right) - \widetilde{V}_\ell\left(y_\ell - \frac{\beta_{2\ell}}{2}\right) \right] - \frac{2\mu_{\ell\ell}\gamma_\ell k_B T_\ell}{\hbar^2} \varepsilon \left[\left(x_\ell - \frac{\beta_{1\ell}}{2}\right) - \left(y_\ell - \frac{\beta_{2\ell}}{2}\right) \right]^2 \right\} \tag{35}$$

where $\mathcal{N}$ is a normalization factor. Since the fast-oscillating terms in the integrals in Eq. (32) contribute only for $\beta_{\ell k} \approx \sqrt{\varepsilon\hbar/M}$, we expand both sides of Eq. (32) up to terms $\mathcal{O}(\varepsilon)$ [2]. Proceeding to a further change of variables: $\beta'_{\ell\ell'} = \beta_{\ell\ell'} + (-1)^\ell \gamma_{\ell'}(x_{\ell'} - y_{\ell'})\varepsilon$, keeping again terms up to $\mathcal{O}(\varepsilon)$, we obtain for the zeroth order term, the normalization factor $\mathcal{N} = (2\pi\hbar\varepsilon)^2 / (\mu_{12}^2 - \mu_1\mu_2)$, and for the first order term the desired equation of motion [2]

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$$\frac{\partial \widetilde{\rho}}{\partial t} = -\sum_\ell \left\{ -i\frac{\hbar}{2m_\ell} \left(\frac{\partial^2}{\partial x_\ell^2} - \frac{\partial^2}{\partial y_\ell^2} \right) + \widetilde{\gamma}_\ell \left(x_\ell - y_\ell \right) \left(\frac{\partial}{\partial x_\ell} - \frac{\partial}{\partial y_\ell} \right) \right.$$

$$+ i\frac{m_\ell \widetilde{\omega}_\ell^2}{2\hbar} \left(x_\ell^2 - y_\ell^2 \right) + \frac{2m_\ell \widetilde{\gamma}_\ell k_B T_\ell}{\hbar^2} \left(x_\ell - y_\ell \right)^2$$

$$+ \sum_{\ell'(\neq \ell)} \left[-i\frac{\hbar \lambda_{22}}{2} \left(\frac{\partial^2}{\partial x_\ell \partial x_{\ell'}} - \frac{\partial^2}{\partial y_\ell \partial y_{\ell'}} \right) + \lambda_{22} m_{\ell'} \widetilde{\gamma}_{\ell'} \left(x_{\ell'} - y_{\ell'} \right) \left(\frac{\partial}{\partial x_\ell} - \frac{\partial}{\partial y_\ell} \right) \right.$$

$$\left. \left. + \lambda_{\ell'\ell} \left(x_{\ell'} \frac{\partial}{\partial x_\ell} + y_{\ell'} \frac{\partial}{\partial y_\ell} \right) + i\frac{\lambda_{\ell'\ell} m_\ell \widetilde{\gamma}_\ell}{\hbar} \left(x_\ell - y_\ell \right) \left(x_{\ell'} + y_{\ell'} \right) + i\frac{\lambda_{11}}{2\hbar} \left(x_\ell x_{\ell'} - y_\ell y_{\ell'} \right) \right] \right\} \widetilde{\rho} \tag{36}$$

where we have defined the effective damping rates

$$\widetilde{\gamma}_\ell = \frac{\gamma_\ell}{1 - \lambda_{22}^2 m_1 m_2}, \tag{37}$$

which increase with increasing coupling strength λ_{22}. The operator equation associated with the above coordinate representation turns out to be

$$\frac{\partial \widetilde{\rho}}{\partial t} = -\frac{i}{\hbar} \left[\mathbf{H}_i, \widetilde{\rho} \right] + \sum_\ell \widetilde{\gamma}_\ell \left[\left(-\frac{i}{\hbar} \left[x_\ell, \{ p_\ell, \widetilde{\rho} \} \right] - \frac{2m_\ell k_B T_\ell}{\hbar^2} \left[x_\ell, \left[x_\ell, \widetilde{\rho} \right] \right] \right) \right. \tag{38}$$

$$\left. -i\frac{m_\ell}{\hbar} \sum_{\ell'(\neq \ell)} \left(\lambda_{22} \left[x_\ell, \{ p_{\ell'}, \widetilde{\rho} \} \right] + \lambda_{\ell'\ell} \left[x_\ell, \{ x_{\ell'}, \widetilde{\rho} \} \right] \right) \right]$$

where the Hamiltonian $\mathbf{H}_i$, is given by

$$\mathbf{H}_i = \frac{1}{2} \sum_\ell \left(\frac{p_\ell^2}{m_\ell} + m_\ell \widetilde{\omega}_\ell^2 x_\ell^2 \right) + \lambda_{11} x_1 x_2 + \lambda_{22} p_1 p_2 + \lambda_{21} p_1 x_2 + \lambda_{12} x_1 p_2. \tag{39}$$

Note that, when turning off the coupling between the two oscillators, we obtain from Eq. (38) two independent dissipative oscillators described by two independent CL models. Therefore, in the case of distinct reservoirs, there is no effective coupling induced between the oscillators.

B. A Common Reservoir

For the case of a common reservoir, we obtain the influence functional

$$\mathcal{F} \left[\{ x_\ell \}, \{ y_\ell \} \right] = \int \mathrm{d}\mathbf{X} \, \mathrm{d}\mathbf{X}' \, \mathrm{d}\mathbf{Y}' \rho_R \left(\mathbf{X}', \mathbf{Y}', 0 \right) \int \mathrm{D}\mathbf{X}(\tau) \, \mathrm{D}\mathbf{Y}(\tau)$$

$$\times \exp \left\{ \frac{i}{\hbar} \left[\mathcal{S}_R \left[\mathbf{X} \right] - \mathcal{S}_R \left[\mathbf{Y} \right] + \sum_\ell \left(\mathcal{S}_{S_\ell/R} \left[x_\ell, \mathbf{X} \right] - \mathcal{S}_{S_\ell/R} \left[y_\ell, \mathbf{Y} \right] \right) \right] \right\}, \tag{40}$$

which differs from that of the CL model for an additional system-reservoir coupling. Assuming identical system-reservoir couplings $C_{1k} = C_{2k} = C_k$, the solution of the influence functional (21), following directly from that in Refs. [33, 34], is given by

$$
\mathcal{F}\left[\{x_\ell\}, \{y_\ell\}\right] = \exp\left(-\frac{1}{\hbar}\sum_\ell \int_0^t d\tau \int_0^\tau dt'\ [x_\ell(\tau) - y_\ell(\tau)]\,\alpha_R(\tau - t')\,[x_\ell(t') - y_\ell(t')]\right)
$$
$$
\times \exp\left(-\frac{i}{\hbar}\sum_\ell \int_0^t d\tau \int_0^\tau dt'\ [x_\ell(\tau) - y_\ell(\tau)]\,\alpha_I(\tau - t')\,[x_\ell(t') + y_\ell(t')]\right),
$$
$$
\tag{41}
$$

where the real and imaginary parts of a function $\alpha(t)$ read

$$
\alpha_R(t) = \sum_k \frac{C_k^2}{2m_k\omega_k} \coth\left(\omega\hbar\beta/2\right)\cos\left[\omega(t)\right], \tag{42a}
$$

$$
\alpha_I(t) = -\sum_k \frac{C_k^2}{2m_k\omega_k} \sin\left[\omega(t)\right]. \tag{42b}
$$

Defining, as in the case of distinct reservoirs, a spectral function

$$
\chi(\omega) = \chi_\ell(\omega) = \sum_k \frac{\pi C_k^2}{2m_k\omega_k}\delta(\omega - \omega_k) \tag{43}
$$

constrained to the range defined by a frequency cutoff Ω^C, and also a damping constant η, we obtain for an Ohmic reservoir

$$
\chi(\omega) = \begin{cases} \eta\omega, & \omega < \Omega^C, \\ 0, & \omega > \Omega^C. \end{cases} \tag{44}
$$

Using exactly the same approximations performed for the case of distinct reservoirs, we

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obtain the influence functional

$$
\mathcal{F}\left[\{x_\ell\},\{y_\ell\}\right]
$$

$$
= \exp\left\{-\frac{i}{\hbar}\left[F\left(\{x_\ell\},\{x'_\ell\},\{y_\ell\},\{y'_\ell\}\right) - \frac{\eta\Omega^C}{\pi}\sum_{\ell,\ell'}\int_0^t \mathrm{d}\tau\ (x_\ell(\tau)x_{\ell'}(\tau) - y_\ell(\tau)y_{\ell'}(\tau))\right.\right.
$$

$$
\left.\left. +\mu\gamma\int_0^t \mathrm{d}\tau\ \sum_\ell\left(\sum_{\ell'}(x_\ell(\tau)\dot{y}_{\ell'}(\tau) - y_\ell(\tau)\dot{x}_{\ell'}(\tau)) + \sum_{\ell'(\neq\ell)}(x_\ell(\tau)\dot{x}_{\ell'}(\tau) - y_\ell(\tau)\dot{y}_{\ell'}(\tau))\right)\right]\right\}
$$

$$
\times \exp\left[-\frac{2\mu\gamma}{\pi\hbar}\sum_{\ell,\ell'}\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}t'\ [x_\ell(\tau) - y_\ell(\tau)]\,[x_{\ell'}(t') - y_{\ell'}(t')]\right.
$$

$$
\left. \times \int_0^{\Omega^C}\mathrm{d}\omega\ \omega\coth\left(\omega\hbar\beta/2\right)\cos\left[\omega(\tau - t')\right]\right] \tag{45}
$$

where

$$
F\left(\{x_\ell\},\{x'_\ell\},\{y_\ell\},\{y'_\ell\}\right) = \frac{\mu\gamma}{2}\sum_\ell\left[\left(x_\ell^2 - x_\ell'^2\right) - \left(y_\ell^2 - y_\ell'^2\right)\right]. \tag{46}
$$

We stress that we have defined, after the assumption $\{C_{\ell k}\} = C_k$, the relaxation constant $\gamma = \eta/2\mu$ for both system-reservoir couplings, which implies immediately that $\{\mu_{\ell\ell}\} = \mu$ and, consequently, $\{m_\ell\} = m$. For the master equation, we obtain

$$
\frac{\partial\widetilde{\rho}}{\partial t} = -\sum_\ell\left\{-i\frac{\hbar}{2m}\left(\frac{\partial^2}{\partial x_\ell^2} - \frac{\partial^2}{\partial y_\ell^2}\right) + i\frac{m\widetilde{\omega}_\ell^2}{2\hbar}\left(x_\ell^2 - y_\ell^2\right)\right.
$$

$$
+ \sum_{\ell'}(x_{\ell'} - y_{\ell'})\left[\frac{2m\widetilde{\gamma}k_BT}{\hbar^2}(x_\ell - y_\ell) + \widetilde{\gamma}\left(\frac{\partial}{\partial x_\ell} - \frac{\partial}{\partial y_\ell}\right)\right]
$$

$$
+ \sum_{\ell'(\neq\ell)}\left[-i\frac{\hbar\lambda_{22}}{2}\left(\frac{\partial^2}{\partial x_\ell\partial x_{\ell'}} - \frac{\partial^2}{\partial y_\ell\partial y_{\ell'}}\right) + \lambda_{\ell'\ell}\left(x_{\ell'}\frac{\partial}{\partial x_\ell} + y_{\ell'}\frac{\partial}{\partial y_\ell}\right)\right.
$$

$$
\left.\left. +\frac{i}{2\hbar}\widetilde{\lambda}_{11}\left(x_\ell x_{\ell'} - y_\ell y_{\ell'}\right) + i\frac{m\widetilde{\gamma}\lambda_{\ell\ell'}}{\hbar}\left(x_\ell + y_\ell\right)\sum_{\ell''}\left(x_{\ell''} - y_{\ell''}\right)\right]\right\}\widetilde{\rho}, \tag{47}
$$

in which the renormalized damping constant $\widetilde{\gamma} = \gamma/\left(1 - m^2\lambda_{22}^2\right)$ and the effective coupling parameter $\widetilde{\lambda}_{11} = \lambda_{11} - 2\Omega^C\eta/\pi$ are considered. As usual, this shift for the effective coupling parameter is small and can be included in λ_{11}. The operator equation associated with the above coordinate representation turns out to be

$$
\frac{\partial\widetilde{\rho}}{\partial t} = -\frac{i}{\hbar}\left[\mathbf{H}_{ii},\widetilde{\rho}\right] - i\frac{m\widetilde{\gamma}}{\hbar}\sum_{\ell,\ell'(\neq\ell)}\lambda_{\ell\ell'}\left[x_{\ell'},\{x_\ell,\widetilde{\rho}\}\right]
$$

$$
-\sum_{\ell,\ell'}\left[i\frac{\widetilde{\gamma}}{\hbar}\left[x_\ell,\{p_{\ell'},\widetilde{\rho}\}\right] + \frac{2m\widetilde{\gamma}k_BT}{\hbar^2}\left[x_\ell,\left[x_{\ell'},\widetilde{\rho}\right]\right]\right] \tag{48}
$$

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where the Hamiltonian $\mathbf{H}_{ii}$, given by

$$\mathbf{H}_{ii} = \frac{1}{2}\sum_{\ell}\left(\frac{p_\ell^2}{m} + m\varpi_\ell^2 x_\ell^2\right) + \lambda_{11}x_1x_2 + \lambda_{22}p_1p_2 + \lambda_{12}x_1p_2 + \lambda_{21}p_1x_2, \qquad (49)$$

contains the renormalized frequencies

$$\varpi_\ell^2 = \widetilde{\omega}_\ell^2 + 2\widetilde{\gamma}\left(\lambda_{12}\delta_{\ell 1} + \lambda_{21}\delta_{\ell 2}\right). \qquad (50)$$

A reservoir-induced coupling between the oscillators

It is worth stressing that, differently from the case of distinct reservoirs presented in Eq. (37), the common reservoir induces an effective coupling between the two oscillators, even when their original interactions $\{\lambda_{\ell\ell'}\}$ are turned off. In fact, with $\lambda_{\ell\ell'} = 0$, the Eq. (48) simplifies to

$$\frac{\partial\widetilde{\rho}}{\partial t} = -\frac{i}{\hbar}\sum_{\ell}[H_{S_\ell}, \widetilde{\rho}] - \sum_{\ell,\ell'}\left[i\frac{\gamma}{\hbar}[x_\ell, \{p_{\ell'}, \widetilde{\rho}\}] + \frac{2m\gamma k_B T}{\hbar^2}[x_\ell, [x_{\ell'}, \widetilde{\rho}]]\right], \qquad (51)$$

giving two independent CL models, for $\ell = \ell'$, apart from the reservoir-induced coupling between the oscillators, for $\ell \neq \ell'$. This effective coupling, for the case analyzed here of an ohmic reservoir in the high-temperature regime, consists therefore of a dissipative and a diffusive term, given by $(\gamma/i\hbar)([x_1, \{p_2, \widetilde{\rho}\}] + [x_2, \{p_1, \widetilde{\rho}\}])$ and $(2m\gamma k_B T/\hbar^2)[x_\ell, [x_{\ell'}, \widetilde{\rho}]]$, respectively. Interestingly, such dissipative and diffusive terms couple together the variables of both oscillators while, evidently, the equivalent terms in the CL model apply to the variables of a single particle.

V.   SOLUTIONS OF THE MASTER EQUATIONS (36) AND (47)

In this section we present the solutions of the master equations governing the dynamics of the coupled dissipative harmonic oscillators in both cases of separate reservoirs and a common one. These solutions enable us to analyze the coherence and decoherence dynamics of quantum superpositions prepared in one of the oscillators of entangled states prepared in both oscillators. Moreover, the form of the solutions presented here enable a complete understanding of the evolution of such states, thus enlarging the perspective of the coherence and decoherence analysis offered by the CL model [4].

A. Distinct Reservoirs

The introduction of the collective (R_ℓ) and relative (r_ℓ) coordinates

$$R_\ell = \frac{x_\ell + y_\ell}{2} \quad , \quad r_\ell = x_\ell - y_\ell \tag{52}$$

enable us to rewrite the master equation (36) in the form

$$
\frac{\partial \widetilde{\rho}}{\partial t} = \sum_\ell \left\{ (-1)^\ell i \frac{\hbar}{m_\ell} \frac{\partial^2}{\partial r_\ell \partial R_\ell} - 2\widetilde{\gamma}_\ell r_\ell \frac{\partial}{\partial r_\ell} + i \frac{m_\ell \omega_\ell^2}{\hbar} R_\ell r_\ell - \frac{D_\ell}{\hbar^2} r_\ell^2 \right.
$$
$$
\left. - \sum_{\ell'(\neq \ell)} \left[\left(\lambda_{\ell\ell'} R_\ell \frac{\partial}{\partial R_{\ell'}} + \Delta_{\ell'} r_{\ell'} \frac{\partial}{\partial r_\ell} \right) + i\hbar \lambda_{22} \frac{\partial^2}{\partial r_\ell \partial R_{\ell'}} - i \frac{\Gamma_\ell}{\hbar} R_{\ell'} r_\ell \right] \right\} \widetilde{\rho}, \tag{53}
$$

where the diffusion coefficients D_ℓ and the effective coupling parameters Γ_ℓ and Δ_ℓ are given by

$$D_\ell = 2m_\ell \widetilde{\gamma}_\ell k_B T_\ell, \tag{54a}$$

$$\Gamma_\ell = \lambda_{11} + 2m_\ell \widetilde{\gamma}_\ell \sum_{\ell'(\neq \ell)} \lambda_{\ell'\ell}, \tag{54b}$$

$$\Delta_\ell = 2\lambda_{22} m_\ell \widetilde{\gamma}_\ell + \sum_{\ell'(\neq \ell)} \lambda_{\ell\ell'}. \tag{54c}$$

By the partial Fourier transform

$$\widetilde{\rho}(\{K_\ell\}, \{r_\ell\}, t) = \frac{1}{2\pi} \left(\prod_\ell \int_{-\infty}^{+\infty} d R_\ell \, e^{-iK_\ell R_\ell} \right) \widetilde{\rho}(\{R_\ell\}, \{r_\ell\}, t) \tag{55}$$

we reduce the second order partial differential equation (53) to the first order one

$$
\frac{\partial \widetilde{\rho}}{\partial t} = -\sum_\ell \left\{ \left(2\widetilde{\gamma}_\ell r_\ell - \frac{\hbar}{m_\ell} K_\ell \right) \frac{\partial}{\partial r_\ell} + \frac{m_\ell \omega_\ell^2}{\hbar} r_\ell \frac{\partial}{\partial K_\ell} + \frac{D_\ell}{\hbar^2} r_\ell^2 \right.
$$
$$
\left. + \sum_{\ell'(\neq \ell)} \left[\left(\frac{\Gamma_{\ell'} r_{\ell'}}{\hbar} - \lambda_{\ell\ell'} K_{\ell'} \right) \frac{\partial}{\partial K_\ell} + \left(\Delta_{\ell'} r_{\ell'} - \hbar \lambda_{22} K_{\ell'} \right) \frac{\partial}{\partial r_\ell} \right] \right\} \widetilde{\rho} \tag{56}
$$

whose solution can be obtained by the method of characteristics [35, 36]. Defining the curves

$$K_\ell = K_\ell(s), \, r_\ell = r_\ell(s), \text{ and } t = t(s) \tag{57}$$

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we obtain, from the partial differential equation (56), a system of coupled ordinary differential equations

$$\frac{\mathrm{d}\,r_\ell}{\mathrm{d}\,s} = -\hbar \sum_{\ell'} \left[\lambda_{22} \left(1 - \delta_{\ell\ell'}\right) + \frac{1}{m_\ell}\delta_{\ell\ell'} \right] K_{\ell'} + 2\widetilde{\gamma}_\ell r_\ell + \left(\Delta_1 r_1 \delta_{\ell 2} + \Delta_2 r_2 \delta_{\ell 1}\right), \tag{58a}$$

$$\frac{\mathrm{d}\,K_\ell}{\mathrm{d}\,s} = \frac{m_\ell \omega_\ell^2}{\hbar} r_\ell + \frac{1}{\hbar} \left[\left(\Gamma_2 r_2 - \hbar \lambda_{12} K_2\right) \delta_{\ell 1} + \left(\Gamma_1 r_1 - \hbar \lambda_{21} K_1\right) \delta_{\ell 2} \right], \tag{58b}$$

$$\frac{\mathrm{d}\,\widetilde{\rho}}{\mathrm{d}\,s} = -\frac{1}{\hbar^2} \sum_\ell D_\ell r_\ell^2 \widetilde{\rho}, \tag{58c}$$

$$\frac{\mathrm{d}\,t}{\mathrm{d}\,s} = 1. \tag{58d}$$

The first four equations of the above system can be expressed, in matrix form, as

$$\frac{\mathrm{d}}{\mathrm{d}\,t}
\begin{pmatrix}
r_1\left(t\right) \\
K_1\left(t\right) \\
r_2\left(t\right) \\
K_2\left(t\right)
\end{pmatrix}
=
\begin{pmatrix}
2\widetilde{\gamma}_1 & -\frac{\hbar}{m_1} & \Delta_2 & -\hbar\lambda_{22} \\
\frac{1}{\hbar}m_1\omega_1^2 & 0 & \frac{1}{\hbar}\Gamma_2 & -\lambda_{12} \\
\Delta_1 & -\hbar\lambda_{22} & 2\widetilde{\gamma}_2 & -\frac{\hbar}{m_2} \\
\frac{\Gamma_1}{\hbar} & -\lambda_{21} & \frac{1}{\hbar}m_2\omega_2^2 & 0
\end{pmatrix}
\begin{pmatrix}
r_1\left(t\right) \\
K_1\left(t\right) \\
r_2\left(t\right) \\
K_2\left(t\right)
\end{pmatrix}, \tag{59}$$

where we shall denote the square matrix by $\mathbf{M}$ and its eigenvalues by Λ_m. Next, we note that the solution of this system of coupled equations can be written in the form

$$\begin{pmatrix}
r_1\left(t\right) \\
K_1\left(t\right) \\
r_2\left(t\right) \\
K_2\left(t\right)
\end{pmatrix}
=
\begin{pmatrix}
\eta_{11} & \eta_{12} & \eta_{13} & \eta_{14} \\
\eta_{21} & \eta_{22} & \eta_{23} & \eta_{24} \\
\eta_{31} & \eta_{32} & \eta_{33} & \eta_{34} \\
\eta_{41} & \eta_{42} & \eta_{43} & \eta_{44}
\end{pmatrix}
\begin{pmatrix}
c_1\left(t\right) \\
c_2\left(t\right) \\
c_3\left(t\right) \\
c_4\left(t\right)
\end{pmatrix}, \tag{60}$$

where the eigenvector $\begin{pmatrix} \eta_{1n} & \eta_{2n} & \eta_{3n} & \eta_{4n} \end{pmatrix}^\top$ is associated with the eigenvalue Λ_n.

Making use of both solutions for $r_\ell\left(t\right)$ and that for the third ordinary differential equation (58c), given by

$$\widetilde{\rho}\left(\{K_\ell\}, \{r_\ell\}, t\right) = B \exp\left[-\frac{1}{\hbar^2} \sum_\ell D_\ell \int \mathrm{d}\,t\, r_\ell^2\left(t\right)\right], \tag{61}$$

we finally obtain the density matrix

$$\widetilde{\rho}\left(\{K_\ell\}, \{r_\ell\}, t\right) = \widetilde{\rho}\left(\{K_\ell'\}, \{r_\ell'\}, 0\right) \mathrm{e}^{-Z\left(\{K_\ell\},\{r_\ell\},t\right)/\hbar^2}, \tag{62}$$

with

$$Z\left(\{K_\ell\},\{r_\ell\},t\right) = \sum_{m,n} c_m\left(t\right) c_n\left(t\right) \frac{D_1 \eta_{1m}\eta_{1n} + D_2\eta_{3m}\eta_{3n}}{\Lambda_m + \Lambda_n}\left(1 - \mathrm{e}^{-(\Lambda_m+\Lambda_n)t}\right),\qquad(63)$$

and $c_m\left(t\right) = c_m\left(0\right)e^{\Lambda_m t}$, where $c_m\left(0\right)$ is determined by the initial conditions.

The next step is to calculate the inverse Fourier transform of Eq. (55). To this end, let us suppose that the coupled harmonic oscillators are prepared at $t = 0$ in the general superposition of Gaussian functions of width $\{\sigma_\ell\}$:

$$\Psi\left(\{x_\ell\},0\right) = \sum_\ell P_\ell \prod_{\ell'} \mathrm{e}^{-[(x_{\ell'}+q_{\ell'\ell})/\sigma_{\ell'}]^2},\qquad(64)$$

which include both separable and entangled states, depending on the choice of the Gaussian centers $q_{1\ell}$ and $q_{2\ell}$ for oscillators 1 and 2. Rewriting the density matrix for this wave function, given by

$$\widetilde{\rho}\left(\{x_\ell\},\{y_\ell\},0\right) = \sum_{\ell,\ell'} P_\ell P_{\ell'} \prod_{\ell''} \mathrm{e}^{-[(x_{\ell''}+q_{\ell''\ell})/\sigma_{\ell''}]^2}\, \mathrm{e}^{-[(y_{\ell''}+q_{\ell''\ell})/\sigma_{\ell''}]^2},\qquad(65)$$

with the collective and relative coordinates defined in Eq. (52), and computing its Fourier transform, we obtain

$$\widetilde{\rho}\left(\{K'_\ell\},\{r'_\ell\},0\right)$$
$$= \frac{1}{4}\sum_{\ell',\ell''} P_{\ell'} P_{\ell''} \prod_\ell \sigma_\ell \exp\left\{-2\left[\left(\frac{\sigma_\ell}{4}K'_\ell\right)^2 - i\left(q_{\ell\ell'} + q_{\ell\ell''}\right)K'_\ell + \left(\frac{q_{\ell\ell'} - q_{\ell\ell''} + r'_\ell}{2\sigma_\ell}\right)^2\right]\right\}.\qquad(66)$$

Therefore, from the results in Eqs. (62) and (66), we obtain the general solution for the transformed master equation (56)

$$\widetilde{\rho}\left(\{K_\ell\},\{r_\ell\},t\right) = \mathcal{R}\left(\{K_\ell\},\{r_\ell\},t\right)\sum_{\ell,\ell'} \Upsilon_{\ell\ell'}$$
$$\times \exp\left\{-\left[\theta^{(1)}_{\ell\ell'}(t)r_1 + \theta^{(2)}_{\ell\ell'}(t)K_1 + \theta^{(3)}_{\ell\ell'}(t)r_2 + \theta^{(4)}_{\ell\ell'}(t)K_2\right]\right\}$$
$$\times \exp\left\{i\left[\widetilde{\theta}^{(1)}_{\ell\ell'}(t)r_1 + \widetilde{\theta}^{(2)}_{\ell\ell'}(t)K_1 + \widetilde{\theta}^{(3)}_{\ell\ell'}(t)r_2 + \widetilde{\theta}^{(4)}_{\ell\ell'}(t)K_2\right]\right\}\qquad(67)$$

which, for $t = 0$, gives the initial condition (65). Assuming that the matrix elements ε_{mn}

follow from the inverse of the matrix composed by η_{mn}, we have defined the functions

$$
\mathcal{R}\left(\{K_\ell\}, \{r_\ell\}, t\right) = \exp\left\{-\left[\Phi_{11}(t)r_1^2 + \Phi_{33}(t)r_2^2 + \Phi_{13}(t)r_1r_2 + \Phi_{12}(t)r_1K_1 + \Phi_{14}(t)r_1K_2\right.\right.
$$
$$
\left.\left. + \Phi_{22}(t)K_1^2 + \Phi_{44}(t)K_2^2 + \Phi_{24}(t)K_1K_2 + \Phi_{23}(t)r_2K_1 + \Phi_{34}(t)r_2K_2\right]\right\}
$$

(68)

$$
\Upsilon_{\ell\ell'} = \frac{1}{4}P_\ell P_{\ell'} \prod_{\ell''} \sigma_{\ell''} \exp\left[-\frac{1}{2}\left(\frac{q_{\ell''\ell} - q_{\ell''\ell'}}{\sigma_{\ell''}}\right)^2\right],
$$

(69a)

$$
\Phi_{kk'}(t) = \sum_{i,j=1(j\geq i)}^{4} (-1)^{i+j} 2^{-\delta_{ij}} \left[\left(\frac{\eta_{1i}\eta_{1j}}{\sigma_1^2} + \frac{\sigma_1^2}{4}\eta_{2i}\eta_{2j} + \frac{\eta_{3i}\eta_{3j}}{\sigma_2^2} + \frac{\sigma_2^2}{4}\eta_{4i}\eta_{4j}\right) e^{-(\Lambda_i+\Lambda_j)t}\right.
$$
$$
\left. + 2\zeta_{ij}\left(1 - e^{-(\Lambda_1+\Lambda_2)t}\right)\right]\left[\varepsilon_{ik}\varepsilon_{jk'} - (1-\delta_{kk'})\varepsilon_{ik'}\varepsilon_{jk}\right],
$$

(69b)

$$
\theta_{\ell\ell'}^{(k)}(t) = (-1)^{\ell+1} \sum_{j=1}^{4}\left(\frac{q_{1\ell} - q_{1\ell'}}{\sigma_1^2}\eta_{1j} + \frac{q_{2\ell} - q_{2\ell'}}{\sigma_2^2}\eta_{3j}\right)\varepsilon_{jk}\, e^{-\Lambda_j t},
$$

(69c)

$$
\widetilde{\theta}_{\ell\ell'}^{(k)}(t) = (-1)^{\ell+1} \sum_{j=1}^{4}\left(\frac{q_{1\ell} + q_{1\ell'}}{2}\eta_{2j} + \frac{q_{2\ell} + q_{2\ell'}}{2}\eta_{4j}\right)\varepsilon_{jk}\, e^{-\Lambda_j t}.
$$

(69d)

The next and final step is to calculate the inverse Fourier transform of Eq. (67). At this point we observe that it is highly recommendable to write the Fourier transformed density matrix $\widetilde{\rho}\left(\{R_\ell\}, \{r_\ell\}, 0\right)$ in its initial normalized form, obtained by a coordinate transformation of Eq. (65):

$$
\widetilde{\rho}\left(\{R_\ell\}, \{r_\ell\}, 0\right) = \left\{\pi\sigma_1\sigma_2\left[1 + \prod_\ell \exp\left[-\frac{1}{2}\left(\frac{q_{\ell 1} - q_{\ell 2}}{\sigma_\ell}\right)^2\right]\right]\right\}^{-1}
$$
$$
\times \sum_{\ell',\ell''} \exp\left[-\sum_\ell \frac{(q_{\ell\ell'} - q_{\ell\ell''} + r_\ell)^2 + (q_{\ell\ell'} + q_{\ell\ell''} + 2R_\ell)^2}{2\sigma_\ell^2}\right].
$$

(70)

The exponentials in Eq. (70) associated with $\ell = \ell'$, represent the diagonal elements of $\widetilde{\rho}\left(\{R_\ell\}, \{r_\ell\}, 0\right)$, which are associated with the probability amplitudes of the system, while exponentials associated with $\ell \neq \ell'$ represent the off-diagonal elements, associated with phase coherence. After the sudden system-reservoir couplings it is expected that the diagonal elements will be dragged to the origin of the coordinates by the dissipative mechanisms, whereas the off-diagonal elements will vanish continuously, owing to the associated noise injection into the system. Therefore, by keeping exactly the form of the above initial density matrix (70) after its time evolution, we directly verify such expected dynamics, simplifying our evaluation of the decoherence effects. From this perspective, we obtain the final solution

$$\widetilde{\rho}\left(\{R_\ell\}, \{r_\ell\}, t\right) = \left\{4\pi\Sigma(t)\left[1 + \prod_\ell \exp\left[-\frac{1}{2}\left(\frac{q_{\ell 1} - q_{\ell 2}}{\sigma_\ell}\right)^2\right]\right]\right\}^{-1}$$

$$\times \sum_{\ell,\ell'} \exp\left\{-\frac{1}{2}\sum_\ell (1 - \delta_{\ell\ell'})\left(\frac{q_{\ell 1} - q_{\ell 2}}{\sigma_\ell}\right)^2 [1 - \Gamma(t)]\right\}$$

$$\times \exp\left[-\frac{1}{2}\Xi_{\ell\ell'}\left(\{R_\ell\}, \{r_\ell\}, t\right) + i\Theta_{\ell\ell'}\left(\{R_\ell\}, \{r_\ell\}, t\right)\right], \qquad (71)$$

where the functions $\Xi_{mn}\left(\{R_\ell\}, \{r_\ell\}, t\right)$, accounting for the dynamics of the diagonal and off-diagonal elements, are given by

$$\Xi_{mn}\left(\{R_\ell\}, \{r_\ell\}, t\right) = \sum_{\ell,\ell'} \left\{2^{\delta_{\ell\ell'}+1}(-1)^{\ell+\ell'}\frac{R_\ell + \theta_{mn}^{(2\ell)}(t)}{\Sigma_{\ell\ell'}(t)}\frac{R_{\ell'} + \theta_{mn}^{(2\ell')}(t)}{\Sigma_{\ell'\ell}(t)}\right.$$

$$\left.+2^{\delta_{\ell\ell'}-1}\frac{r_\ell - [2(q_{\ell 1}(t) - q_{\ell 2}(t))]^{\delta_{\ell\ell'}}}{\sigma_{\ell\ell'}(t)}\frac{r_{\ell'} - [2(q_{\ell' 1}(t) - q_{\ell' 2}(t))]^{\delta_{\ell\ell'}}}{\sigma_{\ell'\ell}(t)}\right\}, \quad (72)$$

whereas the functions

$$\Theta_{mn}\left(\{R_\ell\}, \{r_\ell\}, t\right) = \sum_\ell \left\{2\left[R_\ell + \theta_{mn}^{(2\ell)}(t)\right]\left[(-2)^{\delta_{\ell 1}}\frac{\Phi_{12}(t)r_1 + \Phi_{23}(t)r_2 + \widetilde{\theta}_{mn}^{(2)}(t)}{\Sigma_{\ell 1}^2(t)}\right.\right.$$

$$\left.\left.+ (-2)^{\delta_{\ell 2}}\frac{\Phi_{14}(t)r_1 + \Phi_{34}(t)r_2 + \widetilde{\theta}_{mn}^{(4)}(t)}{\Sigma_{\ell 2}^2(t)}\right] + \theta_{mn}^{(2\ell-1)}(t)r_\ell\right\}, \qquad (73)$$

account only for an oscillatory dynamics. Moreover, we also have, as part of the normalization factor

$$\Sigma(t) = \sqrt{4\Phi_{44}(t)\Phi_{22}(t) - \Phi_{24}^2(t)}, \qquad (74)$$

the functions associated with the widths of the diagonal and off-diagonal Gaussian peaks:

$$\Sigma_{11}(t) = \sqrt{\frac{2\Sigma^2(t)}{\Phi_{44}(t)}} \tag{75a}$$

$$\Sigma_{22}(t) = \sqrt{\frac{2\Sigma^2(t)}{\Phi_{22}(t)}} \tag{75b}$$

$$\Sigma_{12}(t) = \Sigma_{21}(t) = \sqrt{\frac{2\Sigma^2(t)}{\Phi_{24}(t)}} \tag{75c}$$

$$\sigma_{11}(t) = \sqrt{2\Phi_{11}(t) - 2\frac{\Phi_{22}(t)\Phi_{14}^2(t) + \Phi_{44}(t)\Phi_{12}^2(t) - \Phi_{12}(t)\Phi_{14}(t)\Phi_{24}(t)}{\Sigma^2(t)}}, \tag{75d}$$

$$\sigma_{22}(t) = \sqrt{2\Phi_{33}(t) - 2\frac{\Phi_{22}(t)\Phi_{34}^2(t) + \Phi_{44}(t)\Phi_{23}^2(t) - \Phi_{23}(t)\Phi_{34}(t)\Phi_{24}(t)}{\Sigma^2(t)}}, \tag{75e}$$

$$\sigma_{12}(t) = \sigma_{21}(t) = \left[2\Phi_{13}(t) - 2\frac{2\Phi_{22}(t)\Phi_{14}(t) - \Phi_{12}(t)\Phi_{24}(t)}{\Sigma^2(t)}\Phi_{34}(t) \right.$$
$$\left. - 2\frac{2\Phi_{44}(t)\Phi_{12}(t) - \Phi_{14}(t)\Phi_{24}(t)}{\Sigma^2(t)}\Phi_{23}(t) \right]^{1/2}. \tag{75f}$$

Finally, regarding the exponential decay multiplying the off-diagonal elements of the reduced density matrix (71), we obtain the function

$$\Gamma(t) = \left[\sum_\ell \left(\frac{q_{\ell 1} - q_{\ell 2}}{\sigma_\ell} \right)^2 \right]^{-1} \sum_\ell \left[\sigma_\ell^2(t)\vartheta_\ell^2(t) + \sum_{\ell'} 2^{\delta_{\ell\ell'}+1} (-1)^{\ell+\ell'} \frac{\widetilde{\theta}_{12}^{(2\ell)}(t)\widetilde{\theta}_{12}^{(2\ell')}(t)}{\Sigma_{\ell\ell'}^2(t)} \right] \tag{76}$$

where

$$\vartheta_1(t) = \widetilde{\theta}_{1,2}^{(1)}(t) - \frac{[2\Phi_{44}(t)\Phi_{12}(t) - \Phi_{14}(t)\Phi_{24}(t)]\,\widetilde{\theta}_{12}^{(2)}(t) + [2\Phi_{22}(t)\Phi_{14}(t) - \Phi_{12}(t)\Phi_{24}(t)]\,\widetilde{\theta}_{12}^{(4)}(t)}{\Sigma^2(t)}, \tag{77a}$$

$$\vartheta_2(t) = \widetilde{\theta}_{1,2}^{(3)}(t) - \frac{[2\Phi_{44}(t)\Phi_{23}(t) - \Phi_{34}(t)\Phi_{24}(t)]\,\widetilde{\theta}_{12}^{(2)}(t) + [2\Phi_{22}(t)\Phi_{34}(t) - \Phi_{23}(t)\Phi_{24}(t)]\,\widetilde{\theta}_{12}^{(4)}(t)}{\Sigma^2(t)}. \tag{77b}$$

Comparing the initial and the evolved density matrices, given by Eqs. (70) and (71), we verify, as expected, the displacement of both the diagonal and off-diagonal peaks towards the origin of the coordinates, described by Eq. (72). Apart from this displacement towards the origin, the system-reservoir coupling also induces the oscillatory term defined by Eq. (73).

A general decay function of the off-diagonal peaks of the density matrix

From the evolved density matrix in Eq. (71), we also deduce a general expression for the decay $\mathcal{D}(t)$ of the off-diagonal peaks as time goes on, represented by the exponential function

$$\mathcal{D}(t) = \exp\left\{-2\left[1 - \Gamma(t)\right]\sum_{\ell}\left[(q_{\ell 1} - q_{\ell 2})/2\sigma_{\ell}\right]^2\right\}. \tag{78}$$

This decay function offers complete information relative to the decoherence of any initial state of the coupled dissipative oscillators. We anticipate that the same form of the decay function in Eq. (78) will apply to the case of a common reservoir, except for the time-dependent function $\Gamma(t)$, which differs in the two cases as discussed below.

B. A Common Reservoir

The solution of the master equation for the case of a common reservoir is entirely analogous to that of separate reservoirs. From the early assumption of identical system-reservoir couplings $C_{1k} = C_{2k} = C_k$, rendering the same masses m, damping rates γ, and diffusion coefficients D for both oscillators, the master equation, rewritten in terms of the collective and relative coordinates in Eq. (52), is given by

$$\frac{\partial \widetilde{\rho}}{\partial t} = \sum_{\ell}\left[(-1)^{\ell}i\frac{\hbar}{m}\frac{\partial^2}{\partial r_{\ell}\partial R_{\ell}} - 2\widetilde{\gamma}r_{\ell}\frac{\partial}{\partial r_{\ell}} + i\frac{m\omega_{\ell}^2}{\hbar}R_{\ell}r_{\ell} - \frac{D}{\hbar^2}r_{\ell}^2\right.$$
$$\left. - \sum_{\ell'(\neq\ell)}\left(\lambda_{\ell\ell'}R_{\ell}\frac{\partial}{\partial R_{\ell'}} + \Delta_{\ell'}r_{\ell'}\frac{\partial}{\partial r_{\ell}} + i\hbar\lambda_{22}\frac{\partial^2}{\partial r_{\ell}\partial R_{\ell'}} - i\frac{\Gamma_{\ell}}{\hbar}r_{\ell}R_{\ell'}\right)\right]\widetilde{\rho}, \tag{79}$$

where the diffusion coefficients D_{ℓ} and the effective coupling parameters Γ_{ℓ} and Δ_{ℓ} are given by

$$D = 2m\widetilde{\gamma}k_BT, \tag{80a}$$

$$\Gamma_{\ell} = \lambda_{11} + 2m\widetilde{\gamma}\sum_{\ell'(\neq\ell)}\lambda_{\ell'\ell}, \tag{80b}$$

$$\Delta_{\ell} = (\lambda_{21}\delta_{\ell 2} + \lambda_{12}\delta_{\ell 1}) + 2m\widetilde{\gamma}\lambda_{22}. \tag{80c}$$

Following the steps outlined above for the case of distinct reservoirs, leading to the system of coupled ordinary differential equations (81), we now obtain the analogous system

$$\frac{\mathrm{d}\, r_\ell}{\mathrm{d}\, s} = -\hbar \sum_{\ell'} \left[\lambda_{22} \left(1 - \delta_{\ell\ell'} \right) + \frac{1}{m} \delta_{\ell\ell'} \right] K_{\ell'} + 2\widetilde{\gamma} r_\ell + \left(\Delta_1 r_1 \delta_{\ell 2} + \Delta_2 r_2 \delta_{\ell 1} \right), \qquad (81\mathrm{a})$$

$$\frac{\mathrm{d}\, K_\ell}{\mathrm{d}\, s} = \frac{m\omega_\ell^2 r_\ell}{\hbar} + \frac{1}{\hbar} \left[\left(\Gamma_2 r_2 - \hbar \lambda_{12} K_2 \right) \delta_{\ell 1} + \left(\Gamma_1 r_1 - \hbar \lambda_{21} K_1 \right) \delta_{\ell 2} \right], \qquad (81\mathrm{b})$$

$$\frac{\mathrm{d}\, \widetilde{\rho}}{\mathrm{d}\, s} = -\frac{1}{\hbar^2} D \left(r_1 + r_2 \right)^2 \widetilde{\rho}, \qquad (81\mathrm{c})$$

$$\frac{\mathrm{d}\, t}{\mathrm{d}\, s} = 1. \qquad (81\mathrm{d})$$

As before, the first four equations is given in the matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \begin{pmatrix} r_1 \left(t \right) \\ K_1 \left(t \right) \\ r_2 \left(t \right) \\ K_2 \left(t \right) \end{pmatrix} = \begin{pmatrix} 2\widetilde{\gamma} & -\frac{\hbar}{m} & \Delta_2 & -\hbar\lambda_{22} \\ \frac{1}{\hbar} m\omega_1^2 & 0 & \frac{1}{\hbar}\Gamma_2 & -\lambda_{12} \\ \Delta_1 & -\hbar\lambda_{22} & 2\widetilde{\gamma} & -\frac{\hbar}{m} \\ \frac{1}{\hbar}\Gamma_1 & -\lambda_{21} & \frac{1}{\hbar} m\omega_2^2 & 0 \end{pmatrix} \begin{pmatrix} r_1 \left(t \right) \\ K_1 \left(t \right) \\ r_2 \left(t \right) \\ K_2 \left(t \right) \end{pmatrix}. \qquad (82)$$

From here on, all the derivations performed earlier for the case of distinct reservoirs can be followed in exactly the same way, leading to the same structure as the solution given in Eq. (71). The difference is that the elements of the above square matrix lead to values of η_{mn} that differ from those in the case of distinct reservoirs. Moreover, the function $Z \left(\{K_\ell\}, \{r_\ell\}, t \right)$ becomes

$$Z \left(\{K_\ell\}, \{r_\ell\}, t \right) = D \sum_{m,n} c_m \left(t \right) c_n \left(t \right) \frac{\left(\eta_{1m} + \eta_{3m} \right) \left(\eta_{1n} + \eta_{3n} \right)}{\Lambda_m + \Lambda_n} \left(1 - \mathrm{e}^{-(\Lambda_m + \Lambda_n)t} \right), \qquad (83)$$

instead of the expression given in Eq. (63). Therefore, in spite of the solutions to the master equation having the same form, the evolution of the reduced density matrices of the two dissipative oscillators must be quite different for the cases of one common reservoir and two separate ones. We finally point out that the expression for the decay or decoherence $\mathcal{D}(t)$ of the off-diagonal peaks of the initial density operator also has the same structure as that in Eq. (78), the only difference being the time-dependent function $\Gamma(t)$, which differs in the two cases of a common and two separate reservoirs, again due to the values of η_{mn}.

VI. DECOHERENCE IN THE DOUBLE CALDEIRA-LEGGETT MODEL

In this section we analyze the decoherence of three particular entangled states prepared in both oscillators of the network at $t = 0$. These entanglements, derived from Eq. (64)

under the assumption that they all have the same initial mean energy $\langle E \rangle = \langle H_{S_1} + H_{S_2} \rangle$ and distance d between the two peaks in the positional $x_1 x_2$-space, are given by

$$
\Psi^{(1)}\left(\{x_\ell\}, 0\right) = \mathcal{N}^{(1)} \left\{ \exp\left[-\left(x_1 + q_1^{(1)}\right)^2 / \sigma_1^2 \right] \exp\left[-\left(x_2 - q_2^{(1)}\right)^2 / \sigma_1^2 \right] \right.
$$
$$
\left. + \exp\left[-\left(x_1 + q_2^{(1)}\right)^2 / \sigma_1^2 \right] \exp\left[-\left(x_2 - q_1^{(1)}\right)^2 / \sigma_1^2 \right] \right\}, \tag{84a}
$$

$$
\Psi^{(2)}\left(\{x_\ell\}, 0\right) = \mathcal{N}^{(2)} \left\{ \exp\left[-\left(x_1 - q_1^{(2)}\right)^2 / \sigma_2^2 \right] \exp\left[-\left(x_2 - q_2^{(2)}\right)^2 / \sigma_2^2 \right] \right.
$$
$$
\left. + \exp\left[-\left(x_1 - q_2^{(2)}\right)^2 / \sigma_2^2 \right] \exp\left[-\left(x_2 - q_1^{(2)}\right)^2 / \sigma_2^2 \right] \right\}, \tag{84b}
$$

$$
\Psi^{(3)}\left(\{x_\ell\}, 0\right) = \mathcal{N}^{(3)} \left\{ \exp\left[-\left(x_1 + q_1^{(3)}\right)^2 / \sigma_3^2 \right] \exp\left[-\left(x_2 + q_2^{(3)}\right)^2 / \sigma_3^2 \right] \right.
$$
$$
\left. + \exp\left[-\left(x_1 - q_1^{(3)}\right)^2 / \sigma_3^2 \right] \exp\left[-\left(x_2 - q_2^{(3)}\right)^2 / \sigma_3^2 \right] \right\}. \tag{84c}
$$

The choice of equal mean energies $\langle E \rangle$ and distances d follows from the fact that the decoherence process depends on the energy, the distances between the components of the superposition, and the damping rate defining the system-reservoir coupling. It is well-known that the decoherence time varies inversely with the energy, the distance d, and the damping rate. The above states differ from each other only by the position of their peaks in the $x_1 x_2$-space. While both peaks of $\Psi^{(1)}\left(\{x_\ell\}, 0\right)$ ($\Psi^{(2)}\left(\{x_\ell\}, 0\right)$) are positioned in the fourth (first) quadrant of the $x_1 x_2$-space, those of the state $\Psi^{(3)}\left(\{x_\ell\}, 0\right)$ are positioned in the first and third quadrant. Moreover, whereas $\Psi^{(2)}\left(\{x_\ell\}, 0\right)$ is obtained from $\Psi^{(1)}\left(\{x_\ell\}, 0\right)$ by a rotation in $x_1 x_2$-space, $\Psi^{(3)}\left(\{x_\ell\}, 0\right)$ requires, apart from the rotation, a displacement operation over $\Psi^{(1)}\left(\{x_\ell\}, 0\right)$.

We analyze the decoherence time of the above entanglements for the two cases of a common and separate reservoirs. We observe, for comparison, that the analysis of decoherence in Refs. [10, 13, 14], centered on absolute zero reservoirs, shows that the decoherence rate for the case of a common reservoir is attenuated, compared to the case of distinct ones. However, the present analysis is based on the opposite scenario of the high-temperature limit, so that we do not expect to obtain similar results to those in Refs. [10, 13, 14].

As the decoherence rate $\mathcal{D}(t)$ is given by Eq. (78), in Figs. 1, 2, and 3, we plot $\mathcal{D}(t)$ against the scaled time $\gamma_1 t$ for the states $\Psi^{(1)}$, $\Psi^{(2)}$, and $\Psi^{(3)}$, respectively. In parts (a) and (b) of Figs. 1 and 2 we plot the decoherence rate $\mathcal{D}(t)$ for the cases of distinct reservoirs

and a common one, respectively. Adopting unit constants $\hbar, k_B = 1$, masses $(m_1 = m_2 = 1)$, frequency $\omega_1 = 1$ and widths $\sigma_1 = \sigma_2 = 1$, we have set the magnitudes $T_1 = T_2 = 10^3 \hbar \omega_1 / k_B$ and $q_1^{(1)} = q_2^{(2)} = 2q_2^{(1)} = 2q_1^{(2)} = 10\sigma_1$. We have also assumed a regime of parameters where $\gamma_1 = \gamma_2 \ll \{\lambda_{\ell\ell'}\} < \omega_1 = \omega_2/2$, with $\gamma_1/\omega_1 = 10^{-3}$. With these values, the mean energy $\langle E \rangle$ and distance d associated with states (84) becomes $\langle E \rangle \approx 158 \hbar \omega_1$ and $d = 5\sqrt{2}\sigma_1$. To reach the same $\langle E \rangle$ and d for all three states we have assumed that $\sigma_1 = 12\sigma_3$, apart from the relation $q_1^{(3)} = \sqrt{3/2} q_2^{(3)} = \sqrt{15/2} \sigma_1$.

For comparison, the thick solid line in all three figures represents the decoherence time of the Schrödinger-cat-like state

$$\Psi(x, 0) = \mathcal{N} \left\{ \exp\left[-(x+q)^2 / \sigma^2 \right] + \exp\left[-(x-q)^2 / \sigma^2 \right] \right\}, \tag{85}$$

prepared in one of the oscillators, decoupled from the other. This Schrödinger-cat-like state also leads to the same values established above for the mean energy $\langle E \rangle$ and distance d between the two peaks in the x-space. To achieve this, we set the relations $q = 5\sigma_1/\sqrt{2}$ and $\sigma = 6 \times 10^{-2}\sigma_1$. Therefore, the thick solid line describes the decoherence process of a superposition state in the CL problem.

In all three figures, the solid (dashed) and dashed-dotted (dotted) lines describe the decoherence processes when considering distinct reservoirs (a common one) and the coupling between the oscillators given by $\lambda_{11}q_1q_2 + \lambda_{22}p_1p_2$ and $\lambda_{12}q_1p_2 + \lambda_{21}q_2p_1$, respectively, with $\lambda_{\ell\ell'} = 0.1$. Both couplings $\lambda_{11}q_1q_2 + \lambda_{22}p_1p_2$ and $\lambda_{12}q_1p_2 + \lambda_{21}q_2p_1$, when described in terms of the usual annihilation (creation) operators a_1, a_2 ($a_1^\dagger, a_2^\dagger$), correspond to the rotating terms $a_1^\dagger a_2 + a_1 a_2^\dagger$ and the counter-rotating terms $i\left(a_1 a_2 - a_1^\dagger a_2^\dagger \right)$, respectively. We observe that the decay rates of the curves in Figs. 1 and 2 (a and b) are around that associated with the Schrödinger-cat-like state for both cases of a common and distinct reservoirs. Therefore, the case of a common reservoir does not exhibit advantages over that of distinct reservoirs, as demonstrated previously for absolute zero reservoirs [10, 13, 14].

Considering now Fig. 3, we observe that all curves decay faster than those in Figs. 1 and 2. Moreover, the decay rates of the curves associated with the coupling $\lambda_{11}q_1q_2 + \lambda_{22}p_1p_2$ are even faster than those associated with $\lambda_{12}q_1p_2 + \lambda_{21}q_2p_1$. This behavior follows from Eq. (37), which shows that the larger the coupling strength λ_{22}, the larger the effective damping rate $\widetilde{\gamma}_\ell$. The same explanation applies to Fig. 4, where the same curves as in Fig. 1 are plotted for larger strengths $\lambda_{\ell\ell'} = 0.5$; we observe that the decay rates of the curves derived

from the coupling $\lambda_{11}q_1q_2 + \lambda_{22}p_1p_2$ are significantly faster than those for $\lambda_{12}q_1p_2 + \lambda_{21}q_2p_1$, as in Fig. 3. In Figs. 1 and 2, the effect coming from the relation between λ_{22} and $\widetilde{\gamma}_\ell$ is blurred, making the decay rates of the curves associated with the coupling $\lambda_{11}q_1q_2 + \lambda_{22}p_1p_2$ similar to those for $\lambda_{12}q_1p_2 + \lambda_{21}q_2p_1$.

Differently from the works in Refs. [10, 13, 14], where the decay rates were chiefly determined by the assumption of one common or two separate reservoirs, here the difference between the decay rates comes from the different coupling mechanisms between the oscillators, $\lambda_{11}q_1q_2 + \lambda_{22}p_1p_2$ or $\lambda_{12}q_1p_2 + \lambda_{21}q_2p_1$. The reason for this is that Refs. [10, 13, 14] apply to absolute zero reservoirs, where the coupling between the oscillators induced by a common reservoir tends to decrease the decoherence rates. The same effect of coherence control is also achieved in Refs. [10, 13, 14] by assuming that the network oscillators are strongly coupled to each other. Especially in Ref. [10], it is demonstrated that the R-DFSs emerge from situations where the whole network interacts with a common reservoir or when each resonator, strongly coupled to each other, interacts with its own reservoir. The present work, however, applies to the high-temperature regime where, as demonstrated in Figs. 1, 2 and 3, the interaction induced by a common reservoir is completely blurred by the high-temperature effects. From this fact we may expect the high-temperature regime to prevent the emergence of R-DFSs, at least in a network with a small number of oscillators, as in the case at hand.

VII.   CONCLUDING REMARKS

In this study we analyze the double CL model, i.e., the path integral approach to two interacting dissipative harmonic oscillators. We derive and solve the master equations associated with two different situations: when each oscillator is coupled to its own reservoir, and when both oscillators are coupled to a common reservoir. In both cases, the derived master equations consist of two CL models — describing the dissipative mechanism of each oscillator independently — as well as the dynamics arising from the interaction between the two oscillators. However, in the case of a common reservoir, we identify a reservoir-induced coupling between the oscillators, even when the original interaction between them is schwitched off. Such a reservoir-induced coupling, recently pointed out in Ref. [28], encompasses both dissipative and diffusive terms which couple together the variables of both oscillators. These

terms thus account for the energy loss of the oscillators through each other, apart from a joint diffusive process.

The occurrence of such a reservoir-induced coupling between the oscillators was also pointed out in Refs. [10, 13, 14, 20], where networks of dissipative quantum harmonic oscillators are treated through perturbative master equations. In Refs. [10, 13, 14, 20], the occurrence of indirect-decay channels — by which the network oscillators lose excitation through all the other oscillators — is demonstrated in two different situations: when all the (non-interacting or interacting) oscillators are coupled to a common reservoir and even when strongly interacting oscillators are coupled to their own reservoirs.

Regarding the master equation solutions reached in this paper, we stress that their form enables a full comprehension of the evolution of initial states of the network, enlarging the perspective of the coherence and decoherence analysis offered by the CL model [4]. Through these solutions we compute a general expression for the decay rate of the off-diagonal peaks of the density matrix of initial superposition states, which applies to both cases of a common and distinct reservoirs. Such an expression offers complete information about the decoherence of the initial state of the coupled dissipative oscillators. In this regard, we have analyzed, as an application, the decoherence process of particular entanglements in the positional space of both oscillators. The results demonstrated that the coupling induced by the common reservoir does not lead to the collective damping effect mentioned above. The high-temperature regime of validity for our calculations completely blurred such reservoir-induced coupling which at absolute zero works, in general, to delay the decoherence process or even to produce the R-DFSs. However, we find that different interactions between the dissipative oscillators, described by rotating or counter-rotating terms, result in different decay rates of the interference terms of the density matrix. The decay rates associated with the counter-rotating terms of the interaction between the oscillators are significantly faster than those coming from the rotating terms. The reason for this is that the effective damping constants increase with increasing coupling strength related to the counter-rotating terms.

We note that a recent paper addressed the question of the derivation of a master equation for two coupled harmonic oscillators through the influence-functional method of Feynman and Vernon [37]. However, the authors assume a coupling between the oscillators different from that given by Eq. (4), apart from considering only the case of a common reservoir. We finally stress that the present double CL model can be used for the analysis of dissipative

bipolarons, besides other problems in several areas of physics where the CL model has been successfully employed over the last few decades. A more detailed analysis of the development in Ref. [28] is one immediate application of the present study. The effect of temperature on decoherence and the emergence of R-DFSs in the low-temperature regime is also a point worth looking into. Furthermore, the tunneling process of coupled dissipative systems is a phenomenon which may be accounted for by the present work. Together with the achievements in Refs. [10, 13, 14, 20], we believe that the present work furnishes a great deal of material for discussion of the physics of coupled dissipative systems.

VIII. APPENDIX A - DIAGONALIZATION OF THE HAMILTONIAN $H_{S_1+S_2}$

The Hamiltonian modeling the two harmonic oscillators and their mutual interaction, $H_{S_1+S_2}$, as given by Eq. (12) with $V_\ell(q_\ell) = m_\ell \omega_\ell^2 q_\ell^2/2$, can be rewritten in terms of the quantum mechanical creation $a_\ell^\dagger$ and annihilation a_ℓ operators, as

$$H_{S_1+S_2} = \hbar \left[\sum_\ell \omega_\ell \left(a_\ell^\dagger a_\ell + \frac{1}{2} \right) + \left(g_1 a_1 a_2 + g_2 a_1 a_2^\dagger + H.c \right) \right], \tag{86}$$

with the coupling strength

$$g_\ell = \frac{\lambda_{11} - i\lambda_{21} m_1 \omega_1}{\sqrt{m_1 \omega_1 m_2 \omega_2}} + (-1)^\ell \sqrt{m_1 \omega_1 m_2 \omega_2} \left(\lambda_{22} + i \frac{\lambda_{12}}{m_1 \omega_1} \right). \tag{87}$$

The diagonalization of the form (86), easily performed than that in Eq. (12), leads to $H_{S_1+S_2} = \hbar \sum_\ell \Omega_\ell \left(A_\ell^\dagger A_\ell + \frac{1}{2} \right)$, with the normal-mode frequencies,

$$\Omega_\ell^2 = \frac{\omega_1^2 + \omega_2^2}{2} - |g_1|^2 + |g_2|^2 - (-1)^\ell \sqrt{\left(\frac{\omega_1^2 - \omega_2^2}{2} \right)^2 - |g_1|^2 (\omega_1 - \omega_2)^2 + |g_2|^2 (\omega_1 + \omega_2)^2}, \tag{88}$$

and the normal-mode operators

$$A_\ell = \mathcal{N}_\ell \sum_{\ell'} \left[\Delta_{1\ell'}(\Omega_\ell) a_{\ell'}^\dagger + \Delta_{2\ell'}(\Omega_\ell) a_{\ell'} \right], \tag{89}$$

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where

$$\mathcal{N}_{\ell''}^{-2} = \sum_{\ell,\ell'} (-1)^\ell \Delta_{\ell\ell'}^2 (\Omega_{\ell''}), \tag{90a}$$

$$\Delta_{\ell\ell'}(\Omega_{\ell''}) = 2\delta_{\ell'1} \left| g_2 g_{\ell-(-1)^\ell} \right| \omega_2 + \left[\delta_{\ell'2} \left| g_{\ell-(-1)^\ell} \right| - \delta_{\ell1}\delta_{\ell'1} (\Omega_{\ell''} - \omega_2) \right]$$
$$\times \left[|g_1|^2 - |g_2|^2 + (\Omega_{\ell''} - \omega_1)(\Omega_{\ell''} + (-1)^{\ell+\ell'}\omega_2) \right]. \tag{90b}$$

The normal-mode coordinates Q_ℓ and P_ℓ, following from the operators A_ℓ and $A_\ell^\dagger$, are given by

$$Q_\ell = \sqrt{\frac{2\hbar}{\Omega_\ell}} \sum_{\ell'} \mathrm{Re}(a_{\ell\ell'}q_{\ell'} + b_{\ell\ell'}p_{\ell'}), \tag{91a}$$

$$P_\ell = \sqrt{2\hbar\Omega_\ell} \sum_{\ell'} \mathrm{Im}(a_{\ell\ell'}q_{\ell'} + b_{\ell\ell'}p_{\ell'}), \tag{91b}$$

with the coefficients

$$a_{\ell'\ell} = \mathcal{N}_\ell \sqrt{\frac{m_\ell\omega_\ell}{2\hbar}} \sum_{\ell''} \Delta_{\ell''\ell}(\Omega_{\ell'}) \, e^{i(-1)^{\ell''}\phi_\ell}, \tag{92a}$$

$$b_{\ell'\ell} = i\mathcal{N}_\ell \sqrt{\frac{1}{2\hbar m_\ell\omega_\ell}} \sum_{\ell''} (-1)^{\ell''} \Delta_{\ell''\ell}(\Omega_{\ell'}) \, e^{i(-1)^{\ell''}\phi_\ell}, \tag{92b}$$

and phase factors $\phi_\ell = \left[\theta_1 - (-1)^\ell\theta_2 \right]/2$, where the quantities

$$\theta_\ell = \cos^{-1}\left\{ \frac{\xi_{11} + (-1)^\ell\xi_{22}}{[\xi_{11} + (-1)^\ell\xi_{22}]^2 [\xi_{12} + \xi_{21}]^2} \right\}, \tag{93}$$

are defined by the dimensionless strengths

$$\xi_{\ell\ell} = \frac{\lambda_{\ell\ell} \left[\delta_{\ell1} + (m_1 m_2 \omega_1 \omega_2)^{1/2} \delta_{\ell2} \right]}{2 \left[(m_1 m_2 \omega_1 \omega_2)^{1/2} \delta_{\ell1} + \delta_{\ell2} \right]}, \tag{94a}$$

$$\xi_{\ell\ell'} = \frac{\lambda_{\ell\ell'}}{2} \sqrt{\frac{m_\ell\omega_\ell}{m_{\ell'}\omega_{\ell'}}}. \tag{94b}$$

Through the coordinates Q_ℓ and P_ℓ, we finally obtain the diagonalized Hamiltonian

$$\mathcal{H}_{S_1+S_2} = \frac{1}{2} \sum_\ell \left(P_\ell^2 + \Omega_\ell^2 Q_\ell^2 \right). \tag{95}$$

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Figure captions

Fig. 1 Plot of the decay or decoherence function $\mathcal{D}(t)$ against the scaled time $\gamma_1 t$ for the states $\Psi^{(1)}$ in Eq. (84a), considering the cases of (a) distinct reservoirs and (b) a common one. The thick solid line describes the decoherence process of the Schrödinger-cat-like state $\Psi(x,0)$ in Eq. (85), prepared in a single dissipative oscillator. The solid

(dashed) and dashed-dotted (dotted) lines describe the decoherence processes of $\Psi^{(1)}$ when assuming distinct reservoirs (a common one) and the coupling between the oscillators given by $\lambda_{11}q_1q_2 + \lambda_{22}p_1p_2$ and $\lambda_{12}q_1p_2 + \lambda_{21}q_2p_1$, respectively, with $\lambda_{\ell\ell'} = 0.1$.

Fig. 2 Plot of the decay or decoherence function $\mathcal{D}(t)$ against the scaled time $\gamma_1 t$ for the states $\Psi^{(2)}$ in Eq. (84b), considering the cases of (a) distinct reservoirs and (b) a common one. The thick solid line describes the decoherence process of the Schrödinger-cat-like state $\Psi(x,0)$ in Eq. (85), prepared in a single dissipative oscillator. The solid (dashed) and dashed-dotted (dotted) lines describe the decoherence processes of $\Psi^{(2)}$ when assuming distinct reservoirs (a common one) and the coupling between the oscillators given by $\lambda_{11}q_1q_2 + \lambda_{22}p_1p_2$ and $\lambda_{12}q_1p_2 + \lambda_{21}q_2p_1$, respectively, with $\lambda_{\ell\ell'} = 0.1$.

Fig. 3 Plot of the decay or decoherence function $\mathcal{D}(t)$ against the scaled time $\gamma_1 t$ for the states $\Psi^{(3)}$ in Eq. (84c), considering the cases of distinct reservoirs and a common one. The thick solid line describes the decoherence process of the Schrödinger-cat-like state $\Psi(x,0)$ in Eq. (85), prepared in a single dissipative oscillator. The solid (dashed) and dashed-dotted (dotted) lines describe the decoherence processes of $\Psi^{(3)}$ when assuming distinct reservoirs (a common one) and the coupling between the oscillators given by $\lambda_{11}q_1q_2 + \lambda_{22}p_1p_2$ and $\lambda_{12}q_1p_2 + \lambda_{21}q_2p_1$, respectively, with $\lambda_{\ell\ell'} = 0.1$.

Fig. 4 Plot of the decay or decoherence function $\mathcal{D}(t)$ against the scaled time $\gamma_1 t$ for the states $\Psi^{(1)}$ in Eq. (84a), considering the cases of (a) distinct and (b) a common reservoir. The thick solid line describes the decoherence process of the Schrödinger-cat-like state $\Psi(x,0)$ in Eq. (85), prepared in a single dissipative oscillator. The solid (dashed) and dashed-dotted (dotted) lines describe the decoherence processes of $\Psi^{(1)}$ when assuming distinct reservoirs (a common one) and the coupling between the oscillators given by $\lambda_{11}q_1q_2 + \lambda_{22}p_1p_2$ and $\lambda_{12}q_1p_2 + \lambda_{21}q_2p_1$, respectively, with $\lambda_{\ell\ell'} = 0.5$.
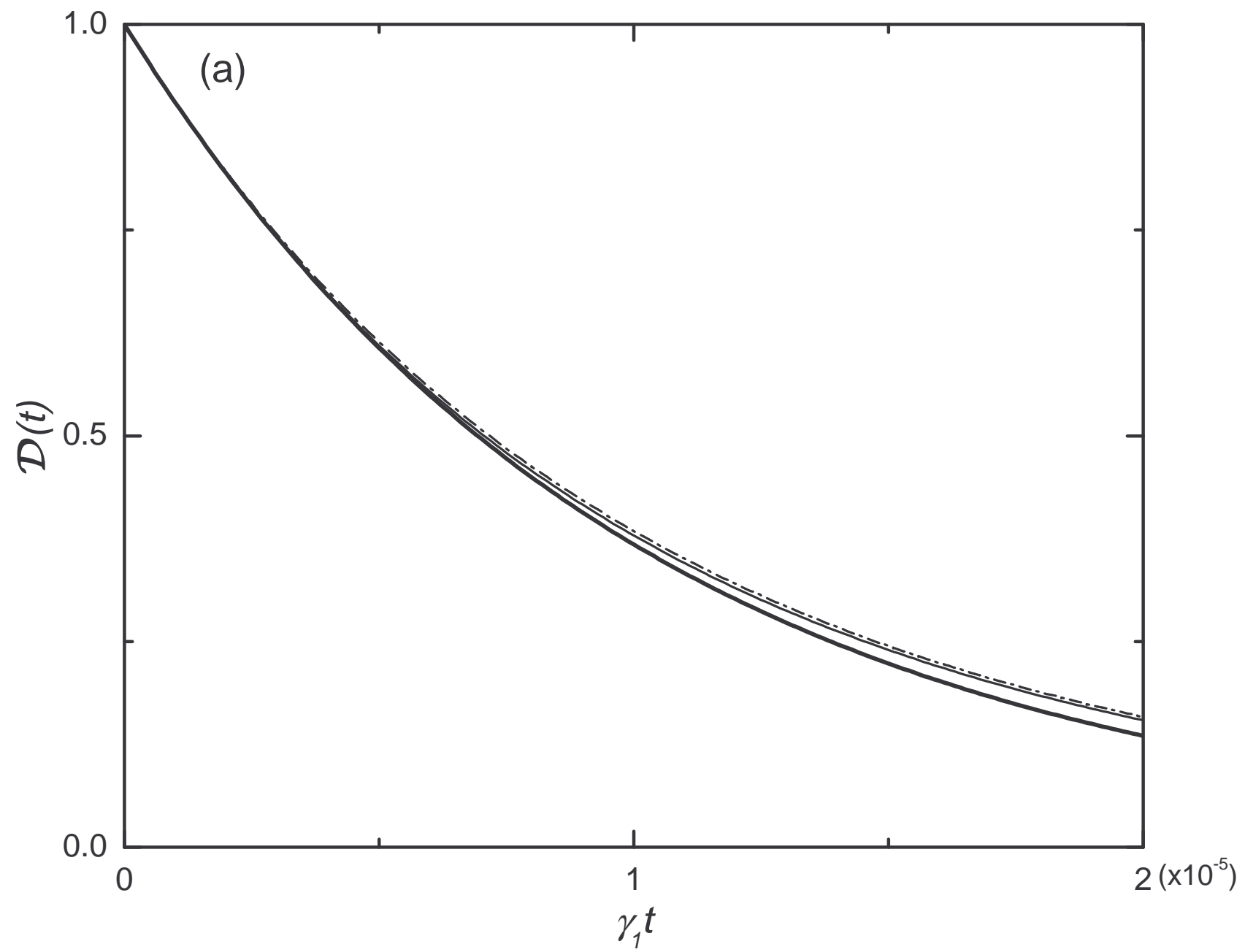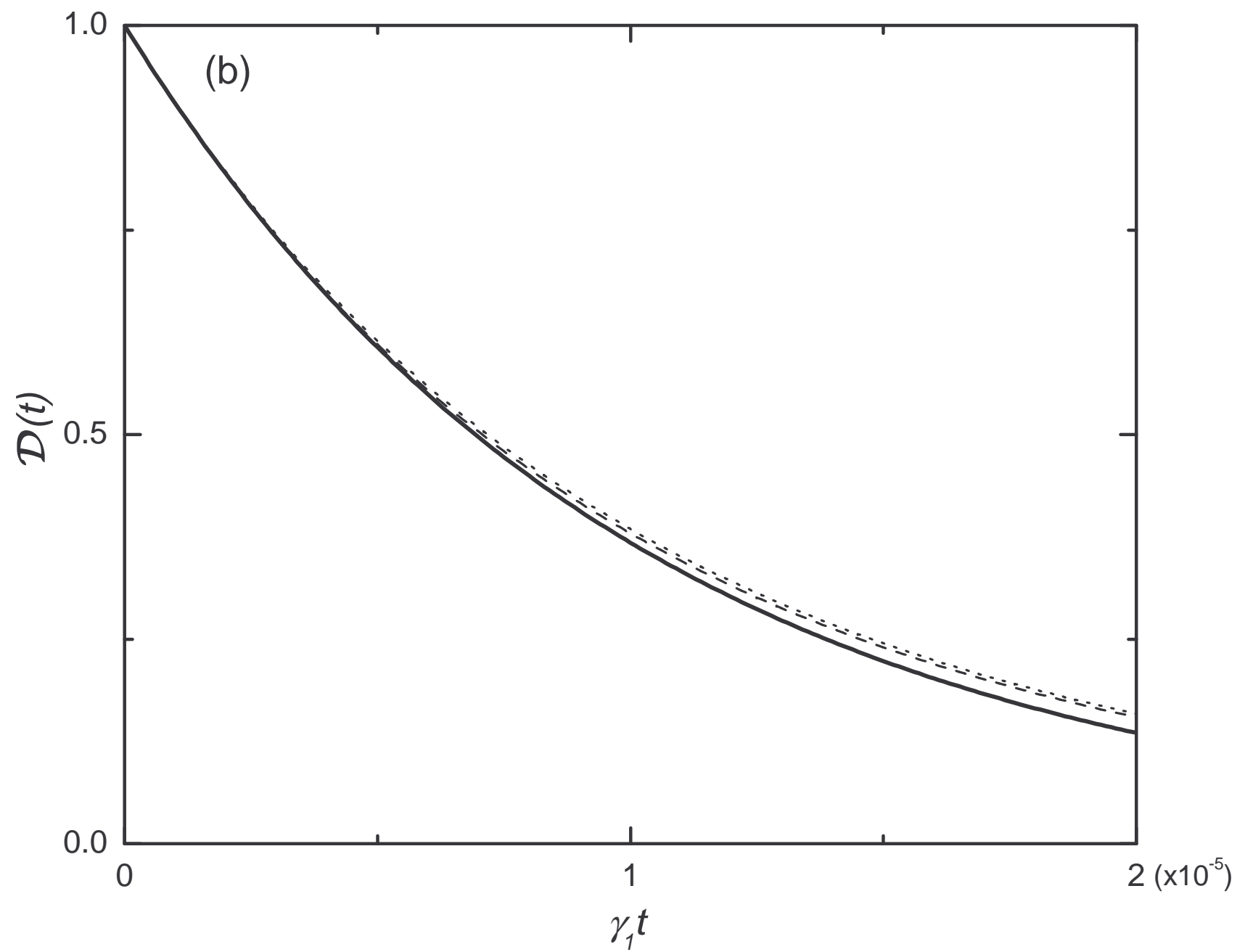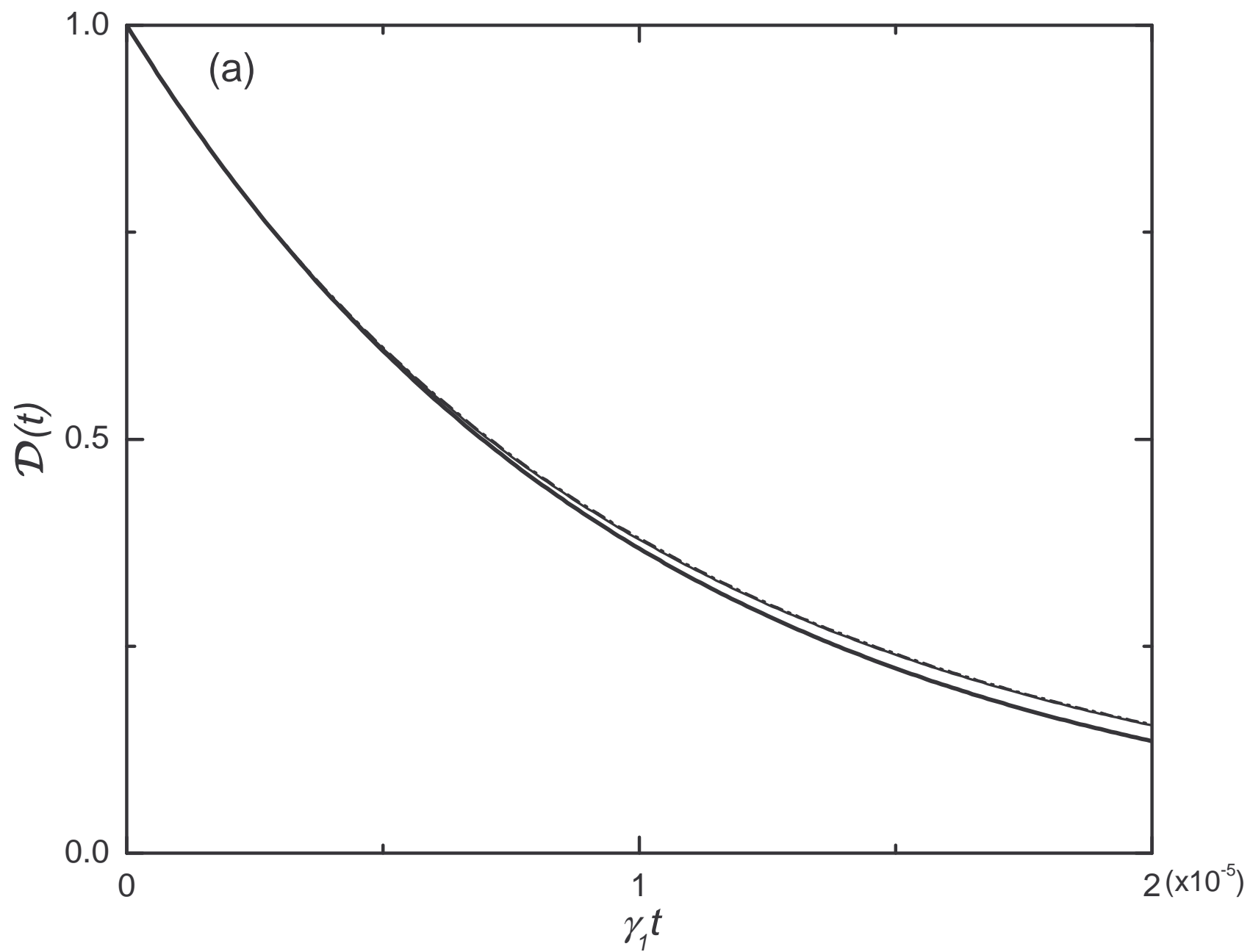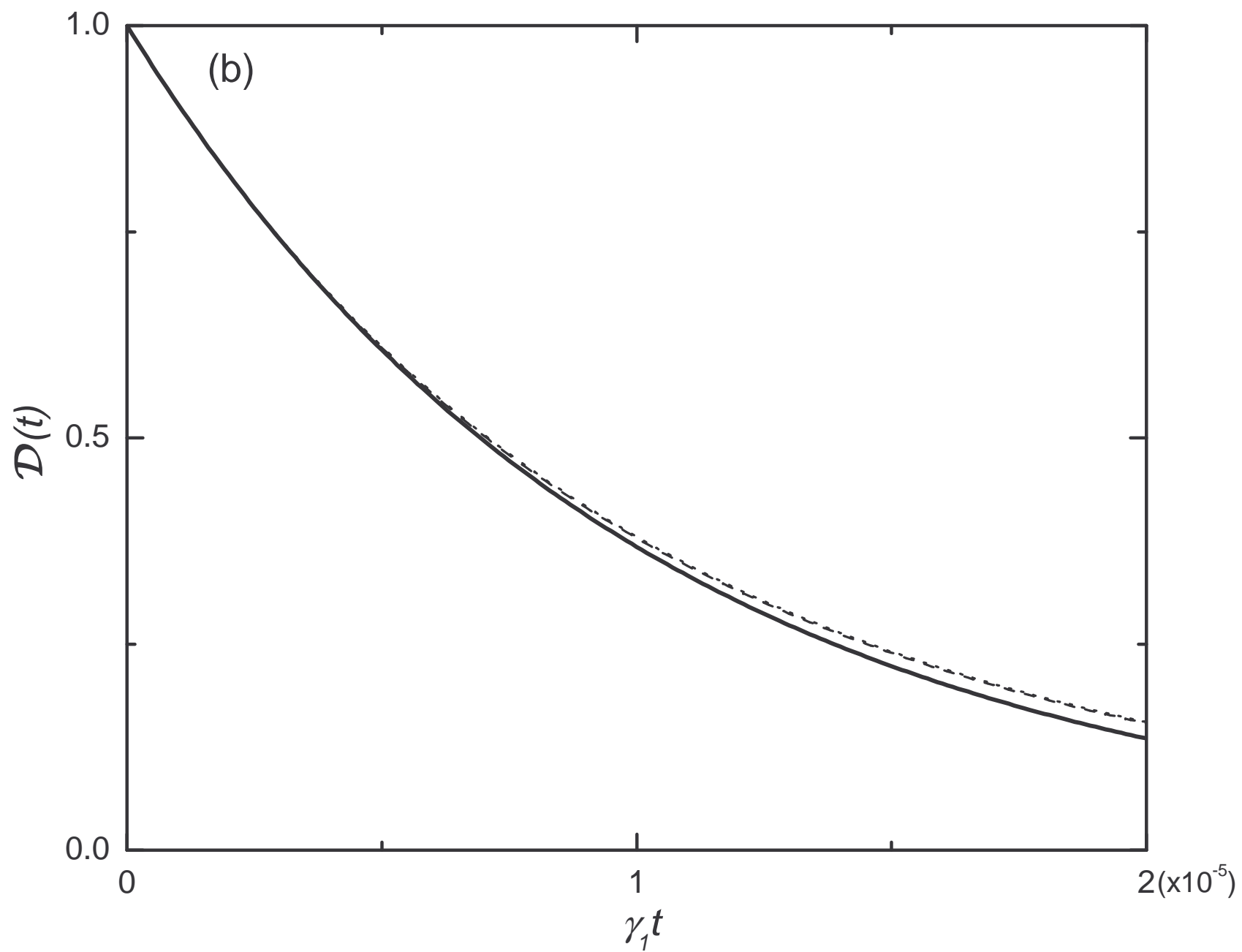
Fig. 1

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